The Ergodic Capacity of Interference Networks

Syed A. Jafar
E-mail : syed@uci.edu

Abstract

We identify the role of equal strength interference links as bottlenecks on the ergodic sum capacity of a network. It is shown that even though there are $K(K-1)$ cross-links, only about $K/2$ disjoint and equal strength interference links suffice to determine the capacity of the network regardless of the strengths of the rest of the cross channels. This scenario is called a *minimal bottleneck state*. It is shown that ergodic interference alignment is capacity optimal for a network in a minimal bottleneck state. The results are applied to large networks. It is shown that large networks are close to bottleneck states with a high probability, so that ergodic interference alignment is close to optimal for large networks. Lastly some interesting examples are provided of parallel channels where separate coding is optimal.

1 Introduction

The capacity of Gaussian interference networks is the holy grail of network information theory. Three vastly different perspectives have been used to approach this challenging problem.

1. **Elemental Networks:** The focus is on small networks - small number of users, finite SNR - in classical settings such as the two user interference channel. This approach is most useful for exploring the microscopic details of optimal coding schemes. The goal is to obtain exact capacity characterizations, or capacity approximations within a constant number of bits. A representative recent result of this approach is the capacity characterization of the 2 user interference channel that is accurate to within 1 bit for all channel parameters [4]. A limitation of the elemental approach is that extensions to more general settings may not be straightforward. For example, it has been shown that extensions to more than 2 users or fading channels require fundamentally new concepts not encountered in the classical two user scenario. Specifically, extensions to more than 2 users involve the new concept of interference alignment [3], while extensions to fading channels have to deal with the inseparability of parallel interference channels[2].

2. **Degrees of Freedom:** The focus is on networks with arbitrary but finite number of users and asymptotically high signal and interference strengths relative to the noise floor at each receiver [6, 3]. This approach offers fundamental insights into the scheduling/medium-access problem by identifying optimal ways of sharing signaling dimensions among competing users. The goal is to obtain capacity approximations within $o(\log(SNR))$, i.e., whose accuracy approaches 100% as SNR approaches infinity. A representative result of this approach is the degrees of freedom characterization of the K user time-varying/frequency-selective interference channel [3]

$$C_{\Sigma} = \frac{K}{2}\log(\text{SNR}) + o(\log(\text{SNR})) \quad (1)$$

where C_Σ is the sum-capacity of the network, K is the number of users and SNR is the signal to noise power ratio. A limitation of this approach is that while the degrees of freedom characterizations are dependent on the number of users through the capacity pre-log, they cannot be directly used to identify scaling laws with the number of users for any finite SNR. For finite SNR, the $o(\log(SNR))$ approximation error term can contain potentially arbitrary dependence on the number of users.

3. **Network Scaling Laws:** The focus is on large networks, with the number of nodes approaching infinity [5, 10]. This approach offers fundamental insights into group routing and scheduling protocols. The goal is to obtain approximations of the logarithm of the capacity within $o(\log(K))$. A recent representative result from this approach is the characterization

$$\log(C_\Sigma) = \log(K) + o(\log(K)) \tag{2}$$

for a dense network of K users [10]. A limitation of this approach is that the dependence on SNR is entirely lost in this perspective.

In this work, we explore the exact ergodic capacity of an interference network, with arbitrary number of users and arbitrary SNR. Combining the achievability results based on the recently proposed ergodic interference alignment scheme [9] with a converse argument based on a notion of bottleneck states introduced in this work, we find that the ergodic setting is surprisingly tractable and allows accurate and simple network capacity characterizations for broad classes of interference networks including many cases commonly considered. For large, dense networks it also provides a scaling law:

$$\lim_{K\to\infty} \text{Prob}\left[\left|\frac{C_\Sigma}{K} - \frac{1}{2}\log(1+2\text{SNR})\right| > \epsilon\right] = 0, \quad \forall\epsilon > 0. \tag{3}$$

In other words,

$$\frac{C_\Sigma}{K} \xrightarrow{\mathbb{P}} \frac{1}{2}\log(1+2\text{SNR}) \tag{4}$$

i.e., for large networks the capacity per user converges in probability to $\frac{1}{2}\log(1+2\text{SNR})$. Note that the dependence on the number of users K as well as SNR is explicit here. For several networks of interest we show that this relationship represents the *exact* non-asymptotic (i.e. arbitrary SNR, arbitrary K) capacity of the network as well.

2 System Model

Consider the general K-user ergodic fading Gaussian interference channel described by the input-output relationship:

$$Y^{[r]}(n) = \sum_{t\in\mathcal{K}} H^{[rt]}(n)e^{j\phi^{[rt]}(n)}X^{[t]}(n) + Z^{[r]}(n), \quad r\in\mathcal{K}, n\in\mathbb{N} \tag{5}$$

where at the n^{th} channel use $Y^{[r]}(n)$ and $Z^{[r]}(n)$ are the received symbol and additive white Gaussian noise (zero mean unit variance circularly symmetric complex Gaussian) seen by receiver r, $X^{[t]}(n)$ is the symbol transmitted from transmitter t, and $H^{[rt]}(n)$ and $\phi^{[rt]}(n)$ are the *strength* and *phase* of the channel between transmitter t and receiver r, $t, r \in \mathcal{K} \triangleq \{1, 2, \cdots, K\}$. All symbols are complex. The power of the transmitted symbols is normalized to 1, i.e., $\mathrm{E}[|X^{[t]}|^2] \leq 1, \forall t \in \mathcal{K}$. The channel phase terms $\phi^{[rt]}(n)$ are i.i.d. and uniform over $[0, 2\pi)$, and independent of the channel strengths. We make the following assumptions regarding the channel strengths.

$$H^{[rt]}(n) = \sqrt{\mathrm{INR}^{[rt]}(n)}, \quad r, t, \in \mathcal{K}, r \neq t, \forall n \in \mathbb{N} \tag{6}$$

$$H^{[kk]}(n) = \sqrt{\mathrm{SNR}}, \ \forall n \in \mathbb{N}, \forall k \in \mathcal{K} \tag{7}$$

The direct SNRs are held constant over time, primarily to avoid non-essential power-control issues. The normalization of SNRs so that all direct links have equal strength is somewhat restrictive but allows for exact capacity results without losing the essence of the problem. It also alleviates fairness concerns always associated with sum capacity. Unless explicitly stated otherwise, no symmetry is assumed in general for the cross-channel strengths $\mathrm{INR}^{[rt]}(n)$. Different cross-channel strengths can follow different distributions and may be correlated in space but are independent identically distributed (i.i.d.) in time. Global channel knowledge is assumed at all transmitters and receivers.

There are K independent messages $W_1, W_2, \cdots, W_K$. Message W_k originates at transmitter k and is intended for receiver k. The codewords are functions of the messages as well as the channel states. The probability of error, achievable rates $R^{[1]}, R^{[r]}, \cdots, R^{[K]}$, and sum capacity of the network C_Σ are defined in the standard Shannon theoretic sense.

We define a *channel state* as the matrix of channel strengths H with elements $H^{[rt]}$, $r, t \in \mathcal{K}$. We use the term "user" to refer to a corresponding transmitter-receiver pair. We use the term capacity only in the sense of the sum capacity of the network. We refer to the fading coefficient $H^{[rt]}(n)e^{j\phi^{[rt]}(n)}$ between any transmitter-receiver pair as a channel or a link interchangeably. Channels $H^{[rt]}$ are called direct channels if $r = t$ and cross-channels if $r \neq t$. We say cross channels $H^{[rt]}(n)$ and $H^{[r't']}(n)$ are disjoint if and only if $\{r, t\} \cap \{r', t'\} = \{\}$, i.e. they do not involve a common user.

3 Background

We start by summarizing the background necessary for this work, which consists of three key ideas - the general idea of interference alignment, the inseparability of interference channels and a specific interference alignment scheme called ergodic interference alignment.

3.1 Interference Alignment

Evolving out of the degrees of freedom study of the X channel [8, 7], the idea of interference alignment was introduced in the context of the interference channel in [3]. It refers to a design of signal vectors so that signals cast overlapping shadows at the receivers where they constitute interference while they remain distinguishable at the receivers where they are desired. The overlap between interferers at each receiver allows interference to be consolidated into a single entity at each receiver. Since the desired signal at each receiver competes with only one effective consolidated interferer, it is possible for every user to access half the channel degrees of freedom - i.e., *everyone gets half the cake* [3].

3.2 Inseparability of Interference Channels

For Gaussian point to point, multiple access and broadcast networks (with no common messages) ergodic capacity is the average of capacities over each channel state (subject to optimal power allocation across channel states). This implies that joint coding across channel states is not needed. However, for interference networks it was shown in [2, 11] that the ergodic capacity is not the average of capacities over each channel state, i.e., parallel interference channels are in general not separable. The following example was provided in [2] to show that even a simple joint coding scheme where interference is treated as noise can achieve not only higher capacity but also more degrees of freedom than the most sophisticated separate coding scheme. Consider a three user interference network, with two parallel states given by channel matrices

$$H = \sqrt{SNR}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad H' = \sqrt{SNR}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \tag{8}$$

It is shown in [2] that separate coding for each channel state can only produce a sum capacity of $2\log(1+3P)$, whereas the capacity with joint coding is $3\log(1+2P)$. Further the joint coding capacity is achieved with the transmitters simply repeating the same symbol over the two sub-channels and the receivers adding their received signals from the two sub-channels. The key to the example is the complementary nature of the two channel matrices, i.e., $H + H' = 2I$ which simply aligns all the interference away from the desired signal.

3.3 Ergodic Interference Alignment

The pairing of complementary matrices exploited in the example above, is generalized to construct an ergodic interference alignment scheme in [9]. Essentially, for every channel matrix H there is a complementary matrix H',

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \quad H' = \begin{bmatrix} h_{11} & -h_{12} & -h_{13} \\ -h_{21} & h_{22} & -h_{23} \\ -h_{31} & -h_{32} & h_{33} \end{bmatrix} \quad H + H' = 2\begin{bmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix} \tag{9}$$

so that if the transmitters repeat the same symbol over the two channel states and the receivers add the outputs from the two channels then all interference is eliminated. With uniform phase distribution the two states are equally likely. Channel state quantization enables a strong typicality argument that with high probability complementary pairs will occur in roughly equal numbers over a long codeword, thus making this alignment possible in an ergodic setting. Based on this approach, the following result is shown in [9].

Theorem 1 *[9] For the K user interference channel of (5), the following ergodic rates are achievable regardless of the cross-channel strengths.*

$$R_k = \frac{1}{2}\log(1 + 2SNR), \quad \forall k \in \mathcal{K}$$

Theorem 1 states that regardless of the number of interferers or the strength of the interferers, each user is able to achieve the same rate that he would achieve if he had the channel to himself, with no interferers, *half the time.* For $K = 2$ users the achievability follows trivially by a channel

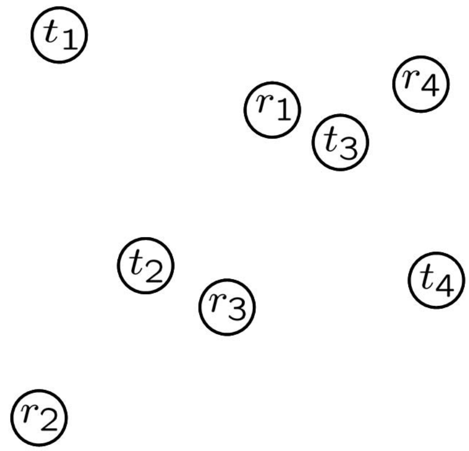

Figure 1: 4 User Interference Network. Distances indicate signal strengths.

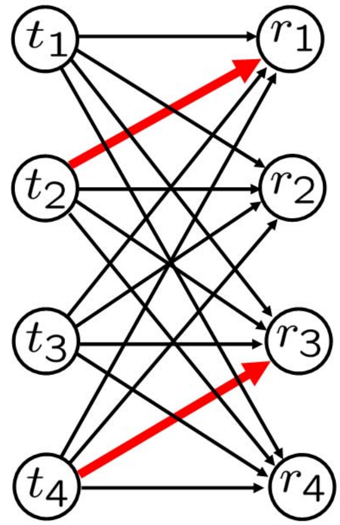

Figure 2: Bottleneck state with bottleneck links highlighted

state-independent TDMA scheme that allows each user to access the channel free from interference for half the time. However, for $K > 2$ users it is the ergodic interference alignment scheme that pairs complementary channel matrices, that is the key to achievability. Theorem 1 constitutes the achievability argument of all the capacity results in this paper. The main contributions of this paper are the converse arguments to show the optimality of ergodic interference alignment.

4 Ergodic Capacity of Interference Networks

Ergodic interference alignment allows every user to achieve (slightly more than) half of their interference-free capacity at any SNR. However, the question remains whether this is the ergodic capacity of the interference network. Consider a $K = 4$ user network example shown in Fig. 1 where the node distances may be translated into relative link strengths. From the figure, it is apparent that receiver 2 only sees weak interference, receiver 4 sees strong interference, while receiver 3 sees both weak and strong interferers. It is known that strong interference can be decoded and cancelled, while weak interference can be treated as noise, in both cases without a significant rate penalty. A user experiencing very strong or very weak interference may be able to achieve close to his full interference-free capacity. For example, in a network where all interferers are very strong, or all interferers are very weak compared to the desired signal strengths, the sum capacity is close to the sum of the users' interference-free capacities. It is clear therefore that achieving only half of the interference-free rate is not necessarily a sum-capacity optimal scheme.

4.1 Bottlenecks

For further insights into the optimality of ergodic interference alignment, consider the generalized degrees of freedom perspective (GDOF) introduced in [4] for the two user interference channel and generalized in [1] to the symmetric K user interference channel. The capacity benefits of weak or strong interference are visible in the GDOF characterization. Indeed it is seen that each user achieves strictly *more* than half his interference-free degrees of freedom, except for two scenarios where $\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})}$ takes values 1 or 1/2. In fact it is known that the network degrees of freedom (α=1) are maximized by each user achieving half his interference-free degrees of freedom. This observation suggests that if the interference is of comparable strength to the desired signal then a scheme that achieves half the interference-free rate for each user may be close to optimal. This intuition forms the foundation for this work. Indeed we establish the role of equal strength interferers as bottlenecks on the network capacity. The following definitions make this notion precise.

- **Bottleneck Link:** The cross-channel $H^{[rt]}(n)$ between receiver r and transmitter t, $t \neq r$, is a bottleneck link if $\text{INR}^{[rt]}(n) = \text{SNR}$, $\forall n \in \mathbb{N}$.
- **Bottleneck State:** The K-user interference network is in a bottleneck state if the network capacity does not depend on the distribution of the non-bottleneck link strengths. In particular, capacity is unchanged if we add more bottleneck links.
- **Reducible Bottleneck States:** A bottleneck state is reducible if relaxing a bottleneck link condition produces another bottleneck state.
- **Minimal Bottleneck State:** A bottleneck state is minimal if there is no other bottleneck state with strictly fewer bottleneck links.

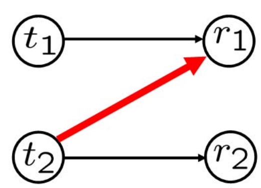

Figure 3: Network of Lemma 1 (only bottleneck and direct links are shown).

Lemma 1 *The two user interference channel with bottleneck link $H^{[12]}$,*

$$Y^{[1]}(n) = \sqrt{SNR}\ e^{j\phi^{[11]}(n)}X^{[1]}(n) + \sqrt{SNR}\ e^{j\phi^{[12]}(n)}X^{[2]}(n) + Z^{[1]}(n) \quad (10)$$

$$Y^{[2]}(n) = \sqrt{INR^{[21]}(n)}\ e^{j\phi^{[21]}(n)}X^{[1]}(n) + \sqrt{SNR}\ e^{j\phi^{[22]}(n)}X^{[2]}(n) + Z^{[2]}(n) \quad (11)$$

has sum capacity

$$C_\Sigma = \log(1 + 2SNR)$$

regardless of the distribution of $INR^{[21]}(n)$ values.

Proof: Achievability follows from Theorem 1 which is based on ergodic interference alignment. The converse is shown as follows. Consider any reliable coding scheme that can achieve arbitrary small probability of error by using appropriately long codewords. Let a genie provide receiver 2 with user 1's message so he can eliminate all the interference due to $X^{[1]}(n)$. Since the coding

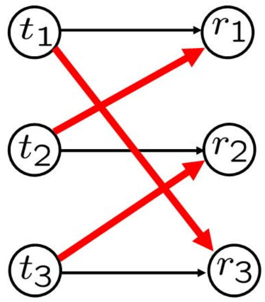

Figure 4: Network of Lemma 2 (bottleneck state) with only direct and bottleneck links shown.

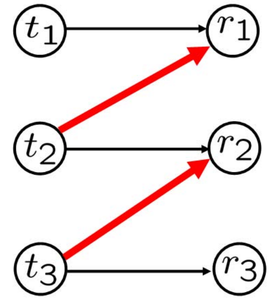

Figure 5: Network of Lemma 3 (not a bottleneck state) with only direct and bottleneck links shown.

scheme is reliable, receiver 1 can also decode his own message and subtract the contribution from $X^{[1]}(n)$ from his received signal. This leaves the two receivers with statistically equivalent signals. Therefore, if receiver 2 can decode message W_2 then so can receiver 1. Since receiver 1 is able to decode both messages, the sum rate of the coding scheme cannot be more than the sum rate capacity of the multiple access channel seen at receiver 1, which is $\log(1 + 2\text{SNR})$. ■

The implication of the term "bottleneck" is clear from the definitions above - the bottleneck links determine the capacity of the network in a bottleneck state regardless of the strengths of the remaining cross-links. For example, in Lemma 1 the link $H^{[12]}$ is a bottleneck link and the link strength of $H^{[21]}$ is irrelevant. Note that we do require the phase of even the non-bottleneck links to vary according to a uniform distribution. This is not needed for the converse but it is needed for the achievability with ergodic interference alignment scheme.

Lemma 2 *The three user interference channel with three bottleneck links $H^{[12]}, H^{[23]}, H^{[31]}$ is in a bottleneck state and has sum capacity*

$$C_\Sigma = \frac{3}{2}\log(1 + 2SNR)$$

regardless of the distribution of the remaining 3 cross-channel strengths.

The network is shown in Fig. 4 with only direct and bottleneck links shown. Note that the non-bottleneck links not shown in the figure are not necessarily zero.

Proof: Achievability again follows from Theorem 1. The converse is easily shown by considering two users at a time. For example, we bound the sum-rate of users $1, 2$ by providing receivers $1, 2$ with transmitter 3's message which allows receivers $1, 2$ to remove interference from transmitter 3 and leaves us with a two user interference channel of the type considered in Lemma 1. This gives us the outer bound $R_1 + R_2 \leq \log(1 + 2\text{SNR})$. Similarly, outer bounds are obtained for the sum rates $R_2 + R_3 \leq \log(1 + 2\text{SNR})$ and $R_1 + R_3 \leq \log(1 + 2\text{SNR})$. Adding these three outer bounds gives us the sum rate capacity outer bound $C_\Sigma \leq \frac{3}{2}\log(1 + 2\text{SNR})$. ■

Lemma 3 *The three user interference channel with two bottleneck links $H^{[12]}, H^{[23]}$ is not in a bottleneck state.*

The network is shown in Fig. 5 with only direct and bottleneck links shown. Note that the non-bottleneck links not shown in the figure are not necessarily zero.

Proof: We prove the result by showing that different values of the non-bottleneck link strength $\text{INR}^{[31]}(n)$ produce different sum-capacity results, thus violating the definition of a bottleneck state. First, setting $\text{INR}^{[31]} = \text{SNR}, \forall n \in \mathbb{N}$ gives us the sum-capacity $C_\Sigma = \frac{3}{2}\log(1+2\text{SNR})$ by Lemma 2. Now alternatively, consider setting all non-bottleneck link strengths $\text{INR}^{[rt]}(n) = 0,\ r \neq t, (r,t) \notin \{(1,2),(2,3)\}, \forall n \in \mathbb{N}$. In this channel let user 2 not transmit at all. Then no interference is seen by receivers 1 and 3 allowing each of them to achieve their single user capacity. In other words, a sum-rate of $2\log(1+\text{SNR})$ is achievable. Since this achievable rate can be higher than $\frac{3}{2}\log(1+2\text{SNR})$, it is clear that the capacity of the 3 user interference channel of Lemma 3 depends on the strengths of the non-bottleneck links, i.e., it is not in a bottleneck state. ■

Similarly, consider the 4 user interference channel example of Fig. 1. As reflected by the node distances in Fig. 1 and highlighted explicitly in Fig. 2 there are two bottleneck links in this 4 user network. By repeating the arguments of Lemma 1 for each disjoint bottleneck link it is easily seen that the 4 user network example of Fig. 1 is in a bottleneck state and its sum capacity is $2\log(1+2\text{SNR})$ regardless of the distribution of the non-bottleneck link strengths.

It is interesting to note that 2 bottleneck links suffice for the 4 user interference channel but not for the 3 user interference channel, where a minimum of 3 bottleneck links is required to put the network in a bottleneck state. This raises the question - what is the minimum number of bottleneck links that can put a K user interference network into a bottleneck state? The bottleneck state with the minimum number of bottleneck links is called a minimal bottleneck state. We characterize the minimal bottleneck state in the following theorem.

Theorem 2 *When K is an even number, a K-user interference channel in a minimal bottleneck state contains exactly $K/2$ bottleneck links and they are mutually disjoint. When K is an odd number, a K-user interference channel in a minimal bottleneck state contains a total of $(K+3)/2$ bottleneck links.*

Theorem 2 shows that very few bottleneck links suffice to limit the capacity of a network. For example, with $K = 10$ users, a minimal bottleneck state needs only 5 bottleneck links out of the total of 90 cross-channels. If 5 bottleneck links are disjoint, then the sum-capacity is determined regardless of the strengths of the remaining 85 cross-channel coefficients. Moreover, there are $\frac{K!}{(K/2)!} = 30,240$ possible distinct minimal bottleneck states.

Proof: Consider first the case that K is even. We first show that there is a bottleneck state with $K/2$ disjoint links. Let the $K/2$ bottleneck links be $H^{[12]}, H^{[34]}, H^{[56]}, \cdots, H^{[K-1,K]}$. Now set the remaining links to zero, which cannot reduce the capacity. This gives us $K/2$ disjoint Z interference channels, each with sum capacity $\log(1+2SNR)$ by Lemma 1. Thus the sum-capacity of this K user interference network cannot be more than $\frac{K}{2}\log(1+2SNR)$ regardless of the strengths of the non-bottleneck links. However, we also know from Theorem 1 that the sum rate $\frac{K}{2}\log(1+2SNR)$ is achievable regardless of the strengths of the cross-links. Thus, the capacity of the K user network with these $K/2$ bottleneck links is $\frac{K}{2}\log(1+2SNR)$ regardless of the strengths of the non-bottleneck links. It is therefore a bottleneck state.

To show that this is a minimal bottleneck state, we construct a proof by contradiction. Suppose there is a bottleneck state with $K/2-1$ bottleneck links. Then, since each bottleneck link can at most involve 2 users, the total number of users involved can at most be $K-2$. Thus, there are at least 2 users that are not associated with any bottleneck link. Setting all non-bottleneck links to zero, we find that the sum-rate $\frac{K-2}{2}\log(1+2SNR)+2\log(1+SNR)$ is achievable. This is because the two users not involved with any interference can each achieve their interference-free capacity $\log(1+SNR)$ while the remainder $K-2$ users can achieve $\frac{1}{2}\log(1+2SNR)$ each by Theorem 1. However, if we add a bottleneck link between the two remaining users, the sum capacity is only $\frac{K}{2}\log(1+2SNR)$. Since the capacity changes by adding a bottleneck link, this cannot be a bottleneck state. The contradiction completes the proof for the case that K is even.

Next, consider the case that K is odd. We first show that there is a bottleneck state with $(K+3)/2$ bottleneck links. Let us first designate $(K-3)/2$ bottleneck links $H^{[45]}, H^{[67]}, \cdots, H^{[K-1,K]}$. This leaves three users who are assigned 3 more bottleneck links $H^{[12]}, H^{[23]}, H^{[31]}$. Once again we set all non-bottleneck links to zero to find the outer bound on sum-capacity $\frac{K}{2}\log(1+2SNR)$. The outer bound for the sum-rate of users 4 through K follows from the disjoint two user Z interference channels associated with each bottleneck link, each of which, from Lemma 1, has sum capacity $\log(1+2SNR)$. The outer bound on the sum-rate of users $1,2,3$ follows from Lemma 2. Once again by Theorem 1, this rate is achievable regardless of the strengths of the non-bottleneck links. Thus, the sum-capacity of the network with a total of $(K-3)/2+3$ bottleneck links is fixed at $\frac{K}{2}\log(1+2SNR)$ regardless of the strengths of the non-bottleneck links. It is, therefore, a bottleneck state.

To show that this is a minimal bottleneck state, we construct a proof by contradiction. Suppose there is a bottleneck state with $\frac{K-3}{2}+2$ bottleneck links. Since $\frac{K-3}{2}$ bottleneck links can only involve at most $K-3$ users, we are left with 2 bottleneck links that can be assigned and 3 users. As shown by Lemma 3, these 3 users cannot be constrained by only two bottleneck links. The contradiction completes the proof. ■

The next theorem shows that all minimal bottleneck states have the same network capacity.

Theorem 3 *For a K-user interference channel in a minimal bottleneck state, the ergodic sum capacity is*

$$C_\Sigma = \frac{K}{2}\log(1+2SNR)$$

Proof: The proof of Theorem 3 is contained in the proof of Theorem 2. ■

Thus, minimal bottleneck states represent classes of interference networks where ergodic interference alignment is exactly optimal.

4.2 Large Networks

Clearly, bottleneck states are of academic interest because they allow an exact capacity characterization. However, the question remains whether these states play a significant role in a network where the link strengths are arbitrary. In particular, how many bottleneck links are there and more importantly, are there enough to put the network in a bottleneck state? Consider, for example an interference network with an even number of users. There are $\frac{K!}{(K/2)!}$ minimal bottleneck states. Is it enough to consider only minimal bottleneck states? Does every bottleneck state contain a minimal bottleneck state hidden behind some redundant bottleneck links ? The following theorem formally answers this question.

Theorem 4 *While every minimal bottleneck state is irreducible, not every irreducible bottleneck state is a minimal bottleneck state.*

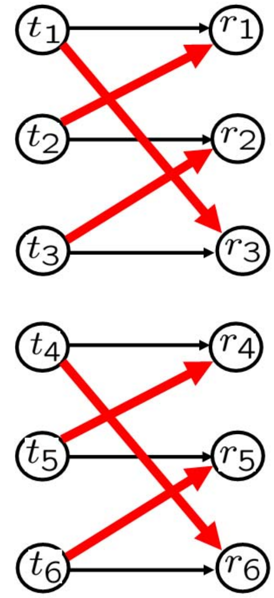

Figure 6: Network of Theorem 4 (only bottleneck and direct links are shown).

Proof: That a minimal bottleneck state is irreducible follows from definition. To show that an irreducible bottleneck state may not be a minimal bottleneck state, we offer an example. Consider the 6 user interference channel in a bottleneck state with 6 bottleneck links as shown in Fig. 6. From the result of Theorem 2 we know that a minimal bottleneck state for a 6 user interference channel requires only 3 bottleneck links. However, from the result of Lemma 3 it is easily seen that one cannot relax any of the 6 bottleneck links in Fig. 6 to obtain another bottleneck state. ■

Thus, there may be many more bottleneck states than minimal bottleneck states. Instead of counting the number of bottleneck states, we proceed to address directly the role of bottleneck states in determining the capacity of a network. We consider a wireless network with a large number of users where cross-channel strengths are determined i.i.d. according to some density function but then they are held fixed for all time,

$$\mathrm{INR}^{[rt]}(n) = \mathrm{INR}^{[rt]}, r \neq t, \forall n \in \mathbb{N}. \tag{12}$$

Note that the direct channel strengths are still all fixed at the same value SNR, and all phase variations are still i.i.d., uniform and independent of the channel strength. Because of the i.i.d. assumption on the density functions, we call such a network a dense network. How likely is it that the network will be in a bottleneck state? Requiring exact equality between INR and SNR makes the probability of a bottleneck link zero, so a network will never be exactly in a bottleneck state. A more useful question is to ask what is the probability that a network will be *close* to a bottleneck state? Closeness here corresponds to the gap between the network capacity outer bound and the sum rate $\frac{K}{2}\log(1+2\mathrm{SNR})$ achievable with ergodic interference alignment. Using a relaxed version of the bottleneck concept, the following result is obtained.

Theorem 5 *For a dense K user interference network, the sum-capacity per user C_Σ/K converges* *in probability to* $\frac{1}{2}\log(1+2SNR)$.

$$\lim_{K\to\infty} Prob\left[\left|\frac{C_\Sigma}{K}-\frac{1}{2}\log(1+2SNR)\right|>\epsilon\right]=0$$

for every $\epsilon>0$.

Thus, ergodic interference alignment achieves the ergodic capacity of a large network in the limit as the number of users approaches infinity. Note that unlike most capacity scaling laws which describe only the exponent of K in the large network limit and are completely insensitive to SNR, this scaling law establishes not only the scaling with number of users K but also the precise SNR dependence of network capacity.

Proof: For the purpose of this theorem, we relax the notion of a bottleneck state as follows. We say a cross-channel is an ϵ-bottleneck link if the sum capacity of the associated two user Z interference channel is within ϵ of $\log(1+2SNR)$, i.e. the sum capacity is less than $\log(1+2SNR)+\epsilon$. The relaxed definition gives us a non-zero probability for a cross-link to be an ϵ-bottleneck link. Let this probability be $\delta>0$. Since the cross-links are generated i.i.d. each link has the same probability δ of being an ϵ-bottleneck. Each ϵ-bottleneck $H^{[rt]}$ introduces the constraint:

$$R^{[r]}+R^{[t]} \leq \log(1+2\text{SNR})+\epsilon \tag{13}$$

Adding these constraints we have:

$$\sum_{r,t\in\mathcal{K},r\neq t} I^{[rt]}(R^{[r]}+R^{[t]}) \leq \left(\sum_{r,t\in\mathcal{K},r\neq t} I^{[rt]}\right)(\log(1+2\text{SNR})+\epsilon) \tag{14}$$

Here $I^{[rt]}$ is an indicator function which takes value 1 when channel $H^{[rt]}$ is a bottleneck link and 0 otherwise. Define sequences of random variables

$$U_K = \frac{1}{K(K-1)}\sum_{r,t\in\mathcal{K},r\neq t} I^{[rt]}\left(\log(1+2\text{SNR})+\epsilon\right) \tag{15}$$

$$V_K = \frac{1}{K(K-1)}\sum_{r,t\in\mathcal{K},r\neq t} I^{[rt]}(R^{[r]}+R^{[t]}) \tag{16}$$

So that by (13)

$$V_K \leq U_K, \quad \forall K\in\mathbb{N} \tag{17}$$

The mean $\overline{U}_K,\overline{V}_K$ and variance $\sigma^2_{U_K},\sigma^2_{V_K}$ are expressed as follows.

$$\overline{U}_K = \delta\left(\log(1+2\text{SNR})+\epsilon\right) \tag{18}$$

$$\overline{V}_K = \delta\frac{2K-1}{K(K-1)}R_\Sigma \tag{19}$$

$$\sigma^2_{U_K} = \frac{\delta(1-\delta)}{K(K-1)}\left[\left(\frac{1}{2}\log(1+2\text{SNR})+\epsilon\right)\right]^2 \tag{20}$$

$$\sigma^2_{V_K} = \frac{\delta(1-\delta)}{K^2(K-1)^2}\sum_{r,t\in\mathcal{K},r\neq t}(R^{[t]}+R^{[r]})^2 \tag{21}$$

$$\leq \frac{4\delta(1-\delta)}{K(K-1)}(\log(1+SNR))^2 \tag{22}$$

where (22) follows from the single user capacity bound

$$R^{[k]} \quad \leq \quad \log(1+SNR), \qquad \forall k \in \mathcal{K} \tag{23}$$

By Chebyshev's inequality

$$\text{Prob}\left[\left|U_K - \overline{U}_K\right| > \frac{\epsilon\delta}{2}\right] \quad \leq \quad \frac{4\sigma^2_{U_K}}{\epsilon^2\delta^2} \tag{24}$$

$$\Rightarrow \text{Prob}\left[U_K \leq \overline{U}_K + \frac{\epsilon\delta}{2}\right] \quad \geq \quad 1 - \frac{4\sigma^2_{U_K}}{\epsilon^2\delta^2} \tag{25}$$

$$\text{Prob}\left[\left|V_K - \overline{V}_K\right| > \frac{\epsilon\delta}{2}\right] \quad \leq \quad \frac{4\sigma^2_{V_K}}{\epsilon^2\delta^2} \tag{26}$$

$$\Rightarrow \text{Prob}\left[V_K \geq \overline{V}_K - \frac{\epsilon\delta}{2}\right] \quad \geq \quad 1 - \frac{4\sigma^2_{V_K}}{\epsilon^2\delta^2} \tag{27}$$

Combining (17), (25) and (27), we have

$$\text{Prob}\left[(U_K \leq \overline{U}_K + \frac{\epsilon\delta}{2}) \text{ AND } (V_K \geq \overline{V}_K - \frac{\epsilon\delta}{2})\right] \quad \geq \quad 1 - \frac{4\sigma^2_{V_K}}{\epsilon^2\delta^2} - \frac{4\sigma^2_{U_K}}{\epsilon^2\delta^2} \tag{28}$$

$$\Rightarrow \text{Prob}\left[\overline{V}_K \leq \overline{U}_K + \epsilon\delta)\right] \quad \geq \quad 1 - \frac{4(\sigma^2_{V_K} + \sigma^2_{U_K})}{\epsilon^2\delta^2} \tag{29}$$

$$\tag{30}$$

Since $\sigma^2_{U_K}, \sigma^2_{V_K} \to 0$ as $K \to \infty$, we have

$$\lim_{K\to\infty} \text{Prob}\left[\frac{2\delta R_\Sigma}{K} \leq \delta\log(1+2\text{SNR}) + 2\epsilon\delta\right] = 1 \tag{31}$$

Since this is true for all achievable rates,

$$\lim_{K\to\infty} \text{Prob}\left[\left|\frac{C_\Sigma}{K} - \frac{1}{2}\log(1+2\text{SNR})\right| > \epsilon\right] = 0 \tag{32}$$

which is the statement of Theorem 5. ■

4.3 Separability of Bottleneck States

In all cases considered so far, the time-variations for the direct links and the bottleneck states have been restricted just to phase variations. (Note that non-bottleneck cross-channels may vary following any distribution). Constant signal strengths with random phases may be appropriate on wireless channels where the propagation is dominated by only one strong path (e.g. the line-of-sight path) whose strength stays relatively constant while its phase may vary significantly due to small scale mobility. When multiple paths are present, however, the phase variations must be associated with channel strength variations as these paths add constructively and destructively even with small mobility. In such scenarios, the time variations of the direct links can be compensated by power control (e.g. channel inversion) which may justify our assumption of constant direct channel SNRs. The same is not true for cross-channels. It is not possible to simultaneously control transmit power so that both the direct links and the cross-links maintain constant strengths. Thus, it is important to consider fading models with varying cross-link strengths. From Theorem 5 we know that in a large (dense) network there are enough bottleneck links to put the network into a bottleneck state when the channel strengths are fixed. If we allow the network signal strengths to vary, then the scenario of Theorem 5 corresponds to a frozen snapshot view for link strengths (while the phases continue to vary). Thus, a fully time varying network may be viewed as a collection of these fixed strength networks in parallel. Since each sub-channel (with fixed signal strengths) is essentially in a bottleneck state, the natural question to explore is the capacity of parallel interference channels, when each sub-channel is in a (possibly different) bottleneck state. Due to inseparability of parallel interference channels, the possibility of joint coding across bottleneck states must be addressed.

Inseparability is already a factor in the time-varying phase model considered in the previous section. The ergodic interference alignment scheme is joint coding scheme. In fact it is the generalization of the very scheme used to prove inseparability of parallel interference channels in [2]. Thus, even with fixed signal strengths, the capacity of the time-varying phase model can only be achieved by joint-coding.

The issue we wish to explore now is the separability of channels across different bottleneck states. The question is answered in part by the results of [11] where the two user fading Z channel is shown to be inseparable [11]. Since phase plays no role on the Z interference channel (it can be normalized so that all phases are zero), it is clear that the inseparability of the Z interference channel is due to the signal strength variations. [11] also provides examples of separable channels, which include scenarios where interference is always very strong, or very weak. For our context, the following theorem elaborates the separability property.

Theorem 6 *Parallel interference channels with channel states corresponding to the same minimal bottleneck state are separable. Parallel interference channels with channel states corresponding to different minimal bottleneck states are in general not separable.*

Note that a given channel state corresponds to fixed channel strengths $H^{[rt]}(n) = H^{[rt]}$ but the phases are i.i.d., time varying with a uniform distribution. Thus, the separability claimed in Theorem 6 only applies to the issue of coding across different states corresponding to different channel strengths. Within the same channel state, joint coding across different phase realizations is always assumed.

Proof: The separability of parallel interference channels with channel states corresponding to the same minimal bottleneck state follows from the definition of bottleneck states where the bottleneck links (fixed strength) determine the capacity regardless of the strengths, or variations thereof, of the remaining links. To prove the inseparability of parallel interference channels with channel states corresponding to different bottleneck states we present the following example (note that while our model and restriction to minimal bottleneck states makes our setting different from [11], the example follows essentially from the observations in [11]).

Consider three parallel 2 user interference channels with states indicated by the following matrices.

$$H_a = \begin{bmatrix} \sqrt{SNR} & \sqrt{SNR} \\ \sqrt{\alpha} & \sqrt{SNR} \end{bmatrix}, \quad H_b = \begin{bmatrix} \sqrt{SNR} & 0 \\ \sqrt{SNR} & \sqrt{SNR} \end{bmatrix}, \quad H_c = \begin{bmatrix} \sqrt{SNR} & 0 \\ \sqrt{SNR} & \sqrt{SNR} \end{bmatrix} \tag{33}$$

While H_b, H_c correspond to the same minimal bottleneck state (and are hence separable among themselves), state H_a is a different minimal bottleneck state. Individually, each state has capacity

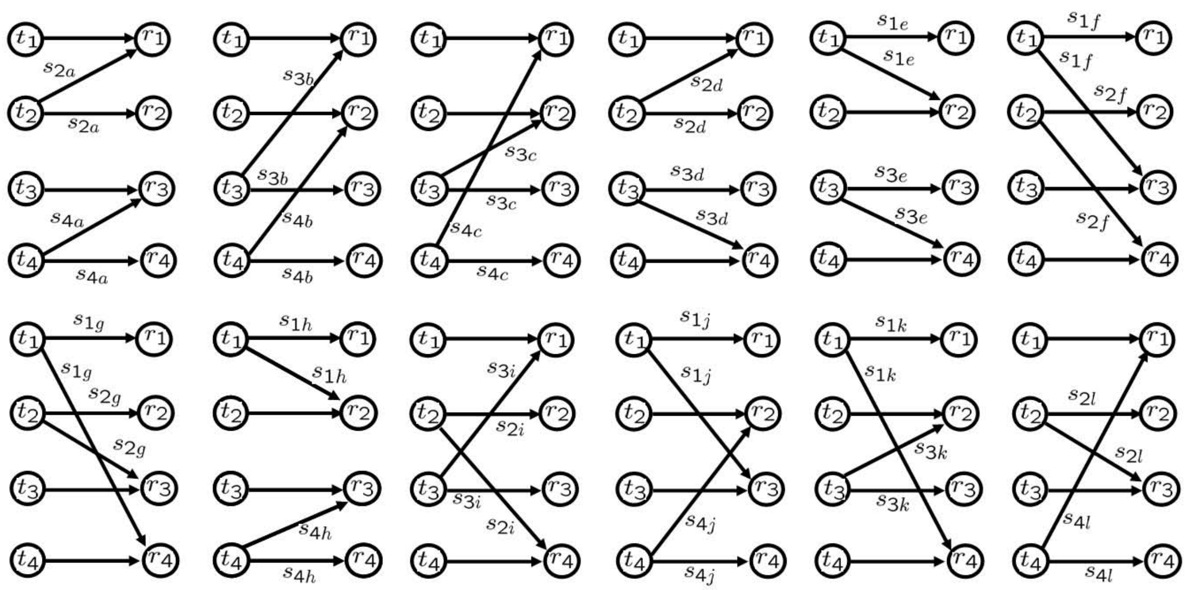

Figure 7: A set of 12 parallel 4 user interference channels that are separable

log(1 + 2SNR), so the sum-capacity with separate coding is 3 log(1 + 2SNR). Now suppose the following condition is satisfied.

$$\log(1+\alpha) \quad \geq \quad 2\log(1+SNR) \tag{34}$$

Then we argue that a sum rate of 4 log(1 + SNR) is achievable. The scheme works as follows. Over state H_a transmitter 1 sends his message W_1 to receiver 2 at rate $2\log(1+SNR)$ which is less than the capacity of the cross-link between transmitter 1 and receiver 2 for state H_a. Receiver 1 and transmitter 2 are inactive in state H_a. Now, in states H_b, H_c the users communicate with their desired receivers as they would in the absence of interference. Receiver 1 does not see any interference and receiver 2 having already decoded transmitter 1's message is able to cancel the interference. Thus the sum-rate achieved is 4 log(1 + SNR) which is larger than that achievable with separate coding. ■

The message from Theorem 6 may be interpreted loosely to mean that similar channel states are separable, while dissimilar ones are not. This is consistent with the results in [11] where the only examples of separable channels correspond to similar channel states (e.g. all weak interference, or all strong interference scenarios). To the best of our knowledge, no separability results are known for channels which are dissimilar, i.e. mixtures of strong and weak interference. Here we make an interesting observation. It turns out (we only consider the case when K is even) that the minimal bottleneck states with all non-bottleneck links set to zero, are separable. The following theorem formally states this result.

Theorem 7 *Consider parallel K user interference channels when K is an even number. Suppose each sub-channel corresponds to a minimal bottleneck state with all non-bottleneck cross-links set* *to zero. Then these parallel channels are separable, i.e., the sum capacity is*

$$C_\Sigma = \frac{K}{2}\log(1+2SNR)$$

which is achieved with separate coding. Equivalently, there is no capacity benefit from joint coding across these parallel interference channels.

An interesting aspect of the proof of Theorem 7 is that none of the standard two user outer bounds can be tight. This is because if we restrict attention to any given set of two users by eliminating all other users' interference, then this set of users is not always connected by a bottleneck link. Thus any converse proved for a fixed pair of users' rates by eliminating interference from all other users would not be tight. Instead we use a new network sum-rate bound that may be of interest by itself.

Also note that the sub-channels in the example of Theorem 7 are the same as the channels used as outer bounds for minimal bottleneck states, i.e. each sub-channel is obtained from a minimal bottleneck state by reducing all non-bottleneck links to zero. If there are no parallel channels, this operation corresponds to giving each receiver the messages of all transmitters to which it is not connected by a bottleneck link. However, it is important to note that in the parallel scenario, setting non-bottleneck links to zero for all sub-channels is not the same as providing some messages to some receivers and it may actually *decrease* capacity. Specifically, providing transmitter t's message to receiver r would correspond to removing interference at receiver r that is due to transmitter t, in *every* sub-channel. This is not the case for the example of Theorem 7 where e.g. receiver 1 sees interference from transmitter 2 in sub-channel a but the interference from the same transmitter is removed in sub-channel b. However, the example is interesting in its own right because it represents a mixed scenario that is separable.

Proof: We present the proof for the $K = 4$ example illustrated in Fig. 7 where each sub-channel is made of $K/2$ disjoint Z channels with bottleneck links. It is easily seen that the proof extends to any even value of K.

We set all phases to zero. There is no loss of generality in this assumption because in each Z channel the transmitters and receivers, who have full channel knowledge by assumption, can always normalize all channel phases.

Next, we introduce noise correlations. We assume that any pair of users connected by a bottleneck link have the same noise at their receivers. Specifically, if $H^{[rt]}, r \neq t$ is a bottleneck link in a minimal bottleneck state, then we assume:

$$Z^{[r]}(n) = Z^{[t]}(n)$$

Since the capacity of the interference channel depends only on the noise marginals, the capacity is unchanged by the noise correlations. Note that this assumption means

$$Y^{[r]}(n) = Y^{[t]}(n) + \sqrt{\text{SNR}}\ X^{[r]}(n), \text{ whenever } H^{[rt]}(n) \neq 0, r \neq t.$$

Define the vectors:

$$\begin{aligned}
\mathbf{S}^{[1]} &= \{Y_e^{[1]}, Y_f^{[1]}, Y_g^{[1]}, Y_h^{[1]}, Y_j^{[1]}, Y_k^{[1]}\} \\
\mathbf{S}^{[1]c} &= \{Y_a^{[1]}, Y_b^{[1]}, Y_c^{[1]}, Y_d^{[1]}, Y_i^{[1]}, Y_l^{[1]}\} \\
\mathbf{S}^{[2]} &= \{Y_a^{[2]}, Y_d^{[2]}, Y_f^{[2]}, Y_g^{[2]}, Y_i^{[2]}, Y_l^{[2]}\}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{S}^{[2]c} &= \{Y_b^{[2]}, Y_c^{[2]}, Y_e^{[2]}, Y_h^{[2]}, Y_j^{[2]}, Y_k^{[2]}\} \\
\mathbf{S}^{[3]} &= \{Y_b^{[3]}, Y_c^{[3]}, Y_d^{[3]}, Y_e^{[3]}, Y_i^{[3]}, Y_k^{[3]}\} \\
\mathbf{S}^{[3]c} &= \{Y_a^{[3]}, Y_f^{[3]}, Y_g^{[3]}, Y_h^{[3]}, Y_j^{[3]}, Y_l^{[3]}\} \\
\mathbf{S}^{[4]} &= \{Y_a^{[4]}, Y_b^{[4]}, Y_c^{[4]}, Y_h^{[4]}, Y_j^{[4]}, Y_l^{[4]}\} \\
\mathbf{S}^{[4]c} &= \{Y_d^{[4]}, Y_e^{[4]}, Y_f^{[4]}, Y_g^{[4]}, Y_i^{[4]}, Y_k^{[4]}\} \\
\mathbf{Y}^{[1]} &= \mathbf{S}^{[1]} \cup \mathbf{S}^{[1]c} \\
\mathbf{Y}^{[2]} &= \mathbf{S}^{[2]} \cup \mathbf{S}^{[2]c} \\
\mathbf{Y}^{[3]} &= \mathbf{S}^{[3]} \cup \mathbf{S}^{[3]c} \\
\mathbf{Y}^{[4]} &= \mathbf{S}^{[4]} \cup \mathbf{S}^{[4]c}
\end{aligned}
$$

$\mathbf{X}^{[k]}, \mathbf{Z}^{[k]}$ are defined in a similar fashion. Starting with Fano's inequality, for codewords spanning N channel uses:

$$
\begin{aligned}
N(R_\Sigma - \epsilon_N) &\leq \sum_{k=1}^{4} I(\mathbf{X}^{[k]N}; \mathbf{Y}^{[k]N}) && (35) \\
&= \sum_{k=1}^{4} I(\mathbf{X}^{[k]N}; \mathbf{S}^{[k]N}) + I(\mathbf{X}^{[k]N}; \mathbf{S}^{[k]cN} | \mathbf{S}^{[k]N}) && (36) \\
&= \sum_{k=1}^{4} h(\mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]N} | \mathbf{X}^{[k]N}) + I(\mathbf{X}^{[k]N}; \mathbf{S}^{[k]cN} | \mathbf{S}^{[k]N}) && (37) \\
&= \sum_{k=1}^{4} h(\mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]N} | \mathbf{X}^{[k]N}) + h(\mathbf{S}^{[k]cN} | \mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]cN} | \mathbf{S}^{[k]N} \mathbf{X}^{[k]N}) && (38) \\
&= \sum_{k=1}^{4} h(\mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]N} | \mathbf{X}^{[k]N}) + h(\mathbf{S}^{[k]cN} | \mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]cN} | \mathbf{X}^{[k]N}) && (39)
\end{aligned}
$$

where (39) follows from the following Markov chain relationship that can be readily verified from Figure 7.

$$
\mathbf{S}^{[k]N} \rightarrow \mathbf{X}^{[k]N} \rightarrow \mathbf{S}^{[k]cN}
$$

Evaluating these terms individually, we have:

$$
\begin{aligned}
h(\mathbf{S}^{[1]cN} | \mathbf{X}^{[1]N}) &= h(S_a^{[2]N}, S_b^{[3]N}, S_c^{[4]N}, S_d^{[2]N}, S_i^{[3]N}, S_l^{[4]N} | \mathbf{X}^{[1]N}) && (40) \\
&= h(S_a^{[2]N}, S_b^{[3]N}, S_c^{[4]N}, S_d^{[2]N}, S_i^{[3]N}, S_l^{[4]N}) && (41) \\
&= h(S_a^{[2]N}, S_d^{[2]N}) + h(S_b^{[3]N}, S_i^{[3]N}) + h(S_c^{[4]N}, S_l^{[4]N}) && (42)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
h(\mathbf{S}^{[2]cN} | \mathbf{X}^{[2]N}) &= h(S_b^{[4]N}, S_c^{[3]N}, S_e^{[1]N}, S_h^{[1]N}, S_j^{[4]N}, S_k^{[3]N} | \mathbf{X}^{[2]N}) && (43) \\
&= h(S_e^{[1]N}, S_h^{[1]N}) + h(S_c^{[3]N}, S_k^{[3]N}) + h(S_b^{[4]N}, S_j^{[4]N}) && (44)
\end{aligned}
$$

$$h(\mathbf{S}^{[3]cN}|\mathbf{X}^{[3]N}) = h(S_a^{[4]N}, S_f^{[1]N}, S_g^{[2]N}, S_h^{[4]N}, S_j^{[1]N}, S_l^{[2]N}|\mathbf{X}^{[3]N}) \quad (45)$$
$$= h(S_f^{[1]N}, S_j^{[1]N}) + h(S_g^{[2]N}, S_l^{[2]N}) + h(S_a^{[4]N}, S_h^{[4]N}) \quad (46)$$

$$h(\mathbf{S}^{[4]cN}|\mathbf{X}^{[4]N}) = h(S_d^{[3]N}, S_e^{[3]N}, S_f^{[2]N}, S_g^{[1]N}, S_i^{[2]N}, S_k^{[1]N}|\mathbf{X}^{[4]N}) \quad (47)$$
$$= h(S_g^{[1]N}, S_k^{[1]N}) + h(S_f^{[2]N}, S_i^{[2]N}) + h(S_d^{[3]N}, S_e^{[3]N}) \quad (48)$$

We expand the first term of (39) to match the terms in (42), (44), (46),(48).

$$\begin{aligned}\sum_{k=1}^{4} h(\mathbf{S}^{[k]N}) \leq\ & h(S_e^{[1]N}, S_h^{[1]N}) + h(S_f^{[1]N}, S_j^{[1]N}) + h(S_g^{[1]N}, S_k^{[1]N}) \\ & + h(S_a^{[2]N}, S_d^{[2]N}) + h(S_g^{[2]N}, S_l^{[2]N}) + h(S_f^{[2]N}, S_i^{[2]N}) \\ & + h(S_b^{[3]N}, S_i^{[3]N}) + h(S_c^{[3]N}, S_k^{[3]N}) + h(S_d^{[3]N}, S_e^{[3]N}) \\ & + h(S_c^{[4]N}, S_l^{[4]N}) + h(S_b^{[4]N}, S_j^{[4]N}) + h(S_a^{[4]N}, S_h^{[4]N}) \end{aligned} \quad (49)$$

Substituting from (42), (44), (46),(48) and (49) into (39) we have:

$$\begin{aligned} N(R_\Sigma - \epsilon_N) &\leq \sum_{k=1}^{4} h(\mathbf{S}^{[k]cN}|\mathbf{S}^{[k]N}) - h(\mathbf{S}^{[k]N}|\mathbf{X}^{[k]N}) \\ &= \sum_{k=1}^{4} h(\mathbf{S}^{[k]cN}|\mathbf{S}^{[k]N}) - 6Nh(Z) \\ &\leq \sum_{k=1}^{4} h(\mathbf{S}^{[k]cN}) - 6Nh(Z) \\ &\leq \sum_{k=1}^{4} 6N \log(1 + 2\text{SNR}) \\ &= 24N \log(1 + 2\text{SNR}) \end{aligned}$$

which, in the limit as $N \to \infty$ and $\epsilon_N \to 0$, for the $K = 4$ user case with 12 bottleneck states considered here, gives us the desired outerbound

$$R_\Sigma \leq \frac{K}{2} \log(1 + 2SNR) \text{ per bottleneck state.}$$

which concludes the proof. ■

Remark: Note that if a set of parallel channels is separable, then so is any subset of that set. Thus, Theorem 7 establishes the separability of all subsets as well.

Remark: While Theorem 7 only addresses the case where the number of users K is even, some extensions to odd K are straightforward. For example, the parallel interference channels obtained from the minimal bottleneck states of the 3 user interference channel by setting all non-bottleneck links to zero, can also be shown to be separable. The proof follows from the result of Theorem 7 by considering only two users at a time.

5 Conclusion

Ergodic interference alignment has recently been shown to achieve (slightly more than) half the interference-free capacity for an interference network at any SNR. We explore the optimality of ergodic interference alignment for ergodic capacity of the interference network. The question arises because it is known that strong interference can be cancelled and weak interference can be treated as noise, without significant penalty to the users' rates. Thus, half of the interference-free rate is not necessarily optimal. The main insight offered in this paper is that network capacity is determined not by the weak or strong interferers but by the equal strength interferers. Thus, we identify equal strength interferers as the bottlenecks for network capacity. It is shown that very few bottleneck links suffice to determine the capacity of the network regardless of the strengths of the remaining links. This notion is formalized in the definition of a minimal bottleneck state. As an example, for a 10 user interference network, 5 disjoint bottleneck links determine the capacity regardless of the strengths of the remaining 85 interfering links. This capacity is achieved by ergodic interference alignment. We also consider a large network and show that it is always close to a bottleneck state, so that ergodic interference alignment is close to capacity optimal for large networks. The impact of time-variations in the interference strength is explored next. While interference channels are in general inseparable, i.e., higher capacity can be achieved by joint encoding across channel states, we find examples where separation is optimal. The examples are interesting because they mix strong and weak interference scenarios and also due to a non-trivial converse proof technique.

Several key questions remain open regarding the ergodic capacity of wireless networks. In particular, greater insights are needed into the inseparability of parallel channels. While we found that different bottleneck states are inseparable, the examples of separable states are intriguing. An open question is to find out if separability holds in an average sense across bottleneck states. Finally, the assumptions of perfect channel knowledge and unbounded delays (to enable ergodic behavior) are crucial to this work. The capacity degradation in moving away from these ideal assumptions is an important issue to be explored in future work.

References

[1] Generalized degrees of freedom of the symmetric Gaussian K user interference channel. In *arXiv:cs/0804.4489 [cs.IT]*, 2008.

[2] V. Cadambe and S. Jafar. Multiple access outerbounds and the inseparability of parallel interference channels. In *Submitted to the IEEE Transactions on Information Theory. Preprint available through http://newport.eecs.uci.edu/ syed*, July 2007.

[3] V. Cadambe and S. Jafar. Interference alignment and the degrees of freedom of the K user interference channel. *IEEE Trans. on Information Theory*, 54(8):3425–3441, Aug. 2008.

[4] R. Etkin, D. Tse, and H. Wang. Gaussian interference channel capacity to within one bit. *submitted to IEEE Transactions on Information Theory*, Feb. 2007.

[5] P. Gupta and P. R. Kumar. The capacity of wireless networks. *IEEE Transactions on Information Theory*, 46(2):388–404, August 2000.

[6] S. Jafar and M. Fakhereddin. Degrees of freedom for the mimo interference channel. *IEEE Transactions on Information Theory*, 53(7):2637–2642, July 2007.

[7] S. Jafar and S. Shamai. Degrees of freedom region for the MIMO X channel. *IEEE Trans. on Information Theory*, 54(1):151–170, Jan. 2008.

[8] M. Maddah-Ali, A. Motahari, and A. Khandani. Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis. In *IEEE Trans. on Information Theory*, pages 3457–3470, 2008.

[9] B. Nazer, M. Gastpar, S. Jafar, and S. Vishwanath. Ergodic interference alignment. Jan 2009. arXiv:0901.4379.

[10] A. Ozgur, O. Leveque, and D. Tse. Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks. *IEEE Transactions on Information Theory*, 53(10):3549 – 3572, Oct 2007.

[11] L. Sankar, X. Shang, E. Erkip, and H. Poor. Ergodic fading two-user interference channels: is separability optimal? In *Proc. of 46th Annual Allerton Conf. on Communications, Control and Computing*, 2008.