Dynamic Connectivity in ALOHA Ad Hoc Networks

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Abstract

In a wireless network the set of transmitting nodes changes frequently because of the MAC scheduler and the traffic load. Analyzing the connectivity of such a network using static graphs would lead to pessimistic performance results. In this paper, we consider an ad hoc network with half-duplex radios that uses multihop routing and slotted ALOHA for the network MAC contention, and introduce a random dynamic multi-digraph to model its connectivity. We first provide analytical results about the degree distribution of the graph. Next, defining the path formation time as the minimum time required for a causal path to form between the source and destination on the dynamic graph, we derive the distributional properties of the connection delay using techniques from first passage percolation and epidemic processes. We show that the delay scales linearly with the distance and provide asymptotic results (with respect to time) for the positions of the nodes which are able to receive information from a transmitter located at the origin. We also provide simulation results to support the theoretical results.

I. Introduction

In a multihop ad hoc network, bits, frames or packets are transferred from a source to a destination in a multihop fashion with the help of intermediate nodes. Decoding, storing, and relaying introduces a delay that, measured in time slots, generally exceeds the number of hops. For example, a five-hop route does not guarantee a delay of only five time slots. In a general setting, each node can connect to multiple nodes. So a large number of paths may form between the source and the destination. Each path may have taken a different time to form with the help of different intermediate nodes. Consider a network in which each node wants to transmit to its destination in a multihop fashion. In general in such a network, a relay node queues the packets from other nodes and its own packets and transmits them according to some scheduling algorithm. If one introduces the concept of queues, the analysis of the system becomes extremely complicated because of the intricate spatial and temporal dependencies between various nodes. In this paper we take a different approach. We are concerned only with the physical connections between nodes, i.e., we do not care when a node i transmits a particular packet to a node j (which depends on the scheduler), but we analyze when a (physical) connection (maybe over multiple hops) is formed between the nodes i and j. This delay is a lower bound on the delay with any queueing scheduler in place.

We assume that the nodes are distributed as a Poisson point process (PPP) on the plane. In each time slot, every node decides to transmit or receive using ALOHA. Any transmitting node can connect to a receiving node when a modified, noiseless version of the protocol model criterion introduced in [1] is met. Since at each time instant, the transmit and receive nodes change, the connectivity graph changes dynamically. We analyze the time required for a causal path to form between a source and a destination node. The system model is made precise in Section II.

This problem is similar in flavor to the problem of First-Passage Percolation (FPP) [2]–[4], and the process of dynamic connectivity also resembles a simple epidemic process [5]–[7] on a Euclidean domain. In a spatial epidemic process, an infected individual infects a certain (maybe random) neighboring population, and this process continues until the complete population is infected or the spreading of the disease stops. In the literature cited above, the time of spread of the epidemic is analyzed for different models of disease spread. We draw many ideas from this theory of epidemic process and FPP. The main difference between an epidemic process and the process we consider is that the spreading (of packets) depends on a subset of the population (due to interference) and is not independent from node to node. In [8], the latency for a message to propagate in a sensor network is analyzed using similar tools. They consider a Boolean connectivity model with randomly weighted edges and derive the properties of first-passage paths on the weighted graph. Their model does not consider interference and thus allows the use of Kingman's subadditive ergodic theorem [9] while ours does not. Percolation in signal-to-interference ratio graphs was analyzed in [10] where the nodes are assumed to be full-duplex. In practice, radios do not transmit and receive at the same time (at the same frequency), and hence the instantaneous network graph is always disconnected. Connectivity between nodes far apart occurs because of the dynamic nature of the MAC protocol. In this paper, we first introduce a dynamic graph process to model and analyze connectivity and then derive the properties of this graph process for ALOHA.

In Section II, we introduce the system model. In Section III, we study the connectivity properties of the random geometric graph formed at any time instant. In Section IV, we derive the properties of the delay and the average number of paths between a source and destination and show that the delay increases linearly with increasing source-destination distance or, equivalently, that the propagation speed is constant, i.e., the distance of the farthest nodes to which the origin can connect increases linearly with time.

II. System Model

The location of the wireless nodes (transceivers) is assumed to be a Poisson point process (PPP) ϕ of intensity λ on the plane. We assume that time is slotted and the MAC protocol used is slotted ALOHA. At every time slot each node transmits with probability p. Nodes are half-duplex, and they act as receivers if they are not transmitting. We use an interference-based model to decide if the communication between a transmitter and a receiver is successful in a given time slot: A transmitting node located at x can connect to a receiver located at y if the disk $B(y, \beta\|x-y\|), \beta > 0$, does not contain any other transmitting nodes. $B(x, r)$ denotes a disk of radius r centered around x and $B^c(x,r) = \mathbb{R}^2 \setminus B(x,r)$. β is a system parameter and captures the resilience of the receiver against interference. This is a variant of the protocol model [1] that does not include the power constraint. The standard SIR model of communication can be related to the protocol model easily when there is no fading. A detailed discussion about the protocol model can be found in [11]. We shall use $\mathbf{1}(x \to y, \Delta)$ to represent a random variable that is equal to one if a transmitter at x is able to connect to a receiver y when the transmitting set is Δ, i.e., the interfering set is $\Delta \setminus \{x\}$. We will drop Δ if there is no ambiguity. At any time instant k, we denote the set of transmitters (decided by ALOHA) by $\phi_t(k)$ and the set of receivers by $\phi_r(k)$. So we have $\phi_t(k) \cup \phi_r(k) = \phi$ and $\phi_t(k) \cap \phi_r(k) = \emptyset$, where $\emptyset$ denotes the empty set.

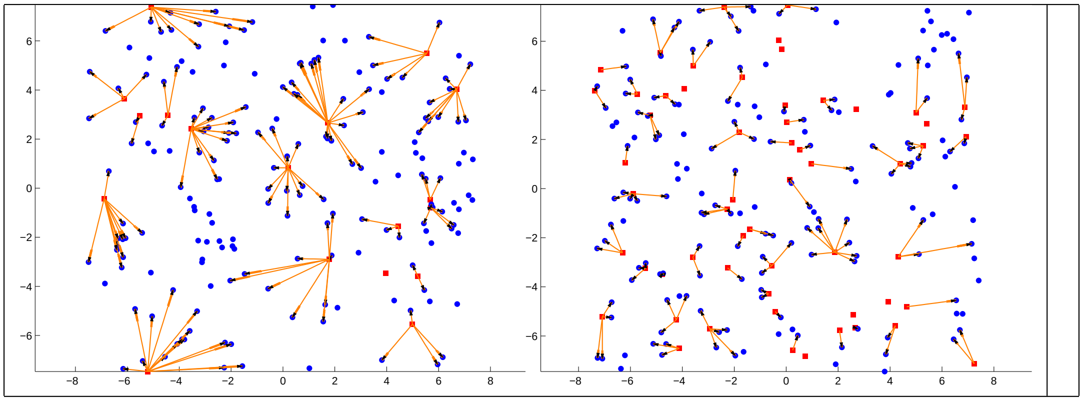

Figure 1. Illustration of a snapshot graph g for $p = 0.2$ (left) and $p = 0.3$ (right) for different realizations of ϕ. The squares represent the transmitters and the circles the receivers.

The connectivity at time k is captured by a directed and weighted random geometric graph $g(k) = (\phi, E_k)$ with vertex set ϕ and edge set

$$E_k = \{(x, y) \colon \; \mathbf{1}\,(x \to y, \phi_t(k)) = 1, x \in \phi_t(k), y \in \phi_r(k)\}\,. \tag{1}$$

See Figure 1 for illustration of $g(0)$ and $g(1)$. Each edge in this graph $g(k)$ is associated with a weight k that represents the time slot in which the edge was formed. Let $G(m, n)$ denote the weighted directed multigraph (multiple edges with different time stamps are allowed between two vertices) formed between times m and $n > m$, i.e.,

$$G(m, n) = \left(\phi, \bigcup_{k=n}^{m} E_k\right).$$

So $G(m, n)$ is the *edge-union* of the graphs $g(k)$, $m \leq k \leq n$. See Figure 2.

Definition 1: A directed path $x_0, e_0, x_1, e_1, \ldots, e_{q-1}, x_q$ between the nodes $x_0 \in \phi$ and $x_q \in \phi$ where $e_i = (x_i, x_{i+1})$ denotes an edge in the multigraph is said to be a *causal path* if the weight of the edges e_i are *strictly increasing* with i.

This means that the edge e_{i-1} was formed before e_i for $0 < i < q$. For the rest of the paper, we always mean causal path when speaking about a path.

We observe that the random graph $g(k)$ is a snapshot of the ALOHA network at time instant k. The random graph process $G(0, m)$ captures the entire connectivity history up to time m. The graph $g(k)$ has the flavor of the interference graph analyzed in [10] where the authors consider only bi-directional links (full-duplex radios). They proved that such a graph percolates with

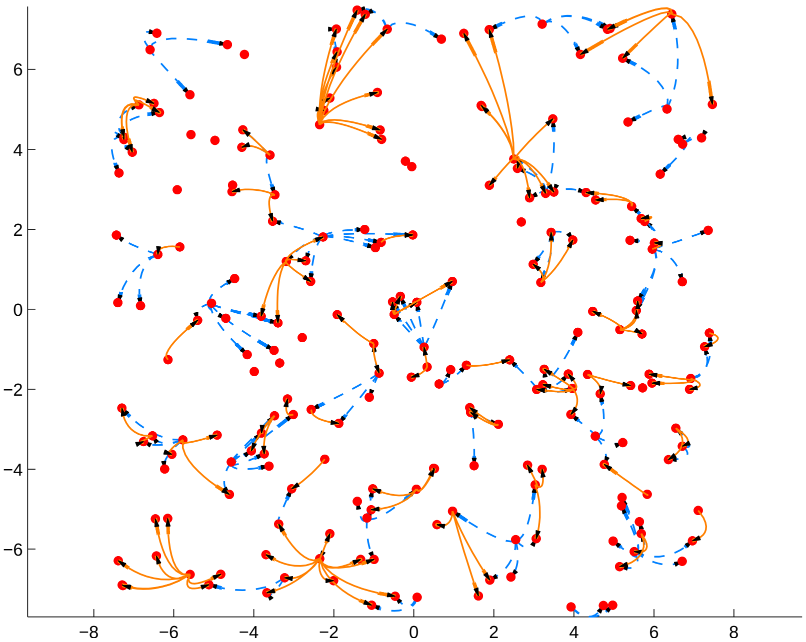

Figure 2. Illustration of $G(0,1)$, $p = 0.2$, $\beta = 1.2$. Dashed line represent edges in $g(0)$ (edges with weight 0) and solid lines represent edges in $g(1)$ (edges with weight 1).

respect to the density of the nodes if the processing gain is high enough. In the graph $G(0,m)$ there is a notion of time and causality, i.e., packets can propagate only on a causal path.

We make the following assumption which we shall use in Section IV. *We assume that the interference at different time instants is independent*. More precisely we assume the following, $\forall m \neq n$ and $\forall\ a, b, c, d \in \phi$,

$$\mathbb{E}[\mathbf{1}_{E_m}((a,b))\,\mathbf{1}_{E_n}((c,d))] = \mathbb{E}[\mathbf{1}_{E_m}((a,b))]\mathbb{E}[\mathbf{1}_{E_n}((c,d))] \tag{2}$$

where the expectation is taken with respect to ALOHA and the point process ϕ. E_m is the edge set defined in (1) and, $\mathbf{1}_{E_m}((a,b))$ is the indicator function of the edge set E_m, which is equal to 1 if and only if the edge (a,b) belongs to E_m.

Assumption (2) is true if $B(b,\beta\|a-b\|) \cap B(d,\beta\|c-d\|) = \emptyset$ or if the node set ϕ is not random (or if we condition on the location of the nodes) since the ALOHA protocol chooses independent transmitter sets across time. In reality interference is not independent in time but almost because of the MAC protocol.

III. PROPERTIES OF THE SNAPSHOT GRAPH $g(k)$

In this section, we will analyze the properties of the random graph $g(k)$. We first observe that the graphs $g(k)$ are identically distributed for all k. So for this section we will drop the time index unless otherwise indicated. g a planar Euclidean graph *even with straight lines* as edges [12, Lemma 2]. In Figure 1, a realization of g is shown for $p = 0.2$ and $p = 0.3$. We first characterize the distribution of the in-degree of a receiver node and the out-degree of a transmit node.

A. Node degree distributions

Let $N_t(x)$ denote the number of receivers a transmitter located at x can connect to, i.e., the out-degree of a transmitting node. Similarly, let $N_r(x)$ denote the number of transmitters that can connect to a receiver at x, i.e., the in-degree of a receiver node. We first calculate the average out-degree of a transmitting node.

Proposition 1: $\mathbb{E}\left[N_t(x)\right] = \frac{1-p}{p}\beta^{-2}$.

Proof: By stationarity of ϕ, we have $N_t(x) \overset{d}{=} N_t(o)$ where $\overset{d}{=}$ stands for equality in distribution. So it is sufficient to consider the out-degree of a transmitter placed at the origin, which is given by $\sum_{x\in\phi_r} \mathbf{1}(o \to x, \phi_t)$. So the average degree is

$$\begin{aligned}
\mathbb{E}[N_t(o)] &= \mathbb{E}\left[\sum_{x\in\phi_r} \mathbf{1}(o \to x, \phi_t)\right] \\
&\overset{(a)}{=} \lambda(1-p)\int_{\mathbb{R}^2} \mathbb{E}_{\phi_t}\left[\mathbf{1}(o \to x, \phi_t)\right] \mathrm{d}x \\
&\overset{(b)}{=} \lambda(1-p)\int_{\mathbb{R}^2} \exp\left(-\lambda p \pi \beta^2 \|x\|^2\right) \mathrm{d}x \\
&= \frac{1-p}{p}\beta^{-2},
\end{aligned}$$

where (a) follows from Campbell's theorem [13] and the independence of ϕ_r and ϕ_t. (b) follows from the fact that $\mathbf{1}(o \to x, \phi_t)$ is equal to one if and only if the ball $B(x, \beta\|x\|)$ does not contain any interferers. ■

We observe that $\mathbb{E}[N_t(x)] \to \infty$ when $p \to 0$. This is because the interference reduces as p becomes smaller. This behavior is a modelling artifact; if the interference vanished, a power constraint would have to be introduced.

Proposition 2: The probability distribution of N_t is given by

$$\mathbb{P}\left(N_t = m\right) = \sum_{k=m}^{\infty} \frac{(-1)^{k+m}}{k!} \left(\frac{1-p}{p}\right)^k V_k(\beta), \tag{3}$$

where $V_k(\beta) = \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \exp\left(-\text{vol}\left(\cup_{i=1}^k B(x_i, \beta\|x_i\|)\right)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k$.

Proof: We provide the complete characterization of N_t using the Laplace transform, given by

$$\begin{aligned}
\mathcal{L}_{N_t}(s) &= \mathbb{E}\left[\exp\left(-sN_t\right)\right] \\
&= \mathbb{E}\left[\exp\left(-s\sum_{x\in\phi_r} \mathbf{1}(o \to x, \phi_t)\right)\right] \\
&\overset{(a)}{=} \mathbb{E}_{\phi_t} \exp\left[-\lambda(1-p)\int_{\mathbb{R}^2} 1 - \exp(-s\mathbf{1}(o \to x, \phi_t))\mathrm{d}x\right] \\
&= \mathbb{E}_{\phi_t} \exp\left[-\lambda(1-p)(1-\exp(-s))\int_{\mathbb{R}^2} \mathbf{1}(o \to x, \phi_t)\mathrm{d}x\right],
\end{aligned} \tag{4}$$

where (a) follows from the probability generating functional of a PPP. The distribution of $\mathbf{1}(o \to x)$ does not change if x is scaled by $\sqrt{\lambda p}$ and the density of ϕ_t is reduced by λp. So, letting ν denote a two dimensional Poisson point process of density 1, we have

$$\mathcal{L}_{N_t}(s) = \mathbb{E}_\nu \exp\left[-\frac{1-p}{p}(1-\exp(-s))\int_{\mathbb{R}^2} \mathbf{1}(o \to x, \nu)\mathrm{d}x\right] \tag{5}$$

Let $a = \frac{1-p}{p}(1-\exp(-s))$. Then

$$\begin{aligned}
\mathcal{L}_{N_t}(s) &= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \mathbb{E}_\nu \left(\int_{\mathbb{R}^2} \mathbf{1}(o \to x, \nu)\mathrm{d}x\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \mathbb{E}_\nu \left(\mathbf{1}(o \to x_1, \nu) \ldots \mathbf{1}(o \to x_k, \nu)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-a)^k}{k!} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \exp\left(-\text{vol}\left(\cup_{i=1}^k B(x_i, \beta\|x_i\|)\right)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k
\end{aligned} \tag{6}$$

By comparison of coefficients (replace e^{-s} with z), we obtain (3). ■

We now derive bounds on the Laplace transform of N_t. Let ν be a PPP of unit intensity on the plane. Define

$$\rho(x,\nu) = \begin{cases} 1 & \text{if } \ \nu \cap B(o,(1+\beta)\|x\|) = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Since $B(x,\beta\|x\|) \subset B(o,(1+\beta)\|x\|)$, we have

$$\mathbf{1}(o \to x, \nu) \geq \rho(x,\nu). \tag{8}$$

So from (5)

$$\mathcal{L}_{N_t}(s) \ \leq \ \mathbb{E}_\nu \exp\left[-a\int_{\mathbb{R}^2} \rho(x,\nu)\mathrm{d}x\right]$$

If the nearest point of ν to the origin is at a distance $\|x\|$, the value of the integral is $\frac{\pi x^2}{(\beta+1)^2}$, and since the first contact distribution of the Poisson point process is Rayleigh [13], we have

$$\begin{aligned} \mathcal{L}_{N_t}(s) \ &\leq \ \int_0^\infty \exp\left[-a\frac{\pi x^2}{(\beta+1)^2}\right]\exp(-\pi x^2)2\pi x\mathrm{d}x \\ &= \ \frac{1}{1+\frac{a}{(\beta+1)^2}} \end{aligned}$$

We can now use the Chernoff bound and the above inequality to obtain the following bound on the CCDF:

$$\mathbb{P}(N_t > k) \leq \frac{e^{ks}}{1+(\beta+1)^{-2}p^{-1}(1-p)\left(1-e^{-s}\right)} \ , \ \forall s > 0$$

the right hand side is minimized for $s = \ln\left((1+k^{-1})/(1+(\beta+1)^2p(1-p)^{-1})\right)$. We have

$$\mathbb{P}(N_t > k) \leq \frac{kp(\beta+1)^2\left(\frac{(k+1)(1-p)}{k(p\beta(\beta\ +2)+1)}\right)^{k+1}}{1-p}$$

A lower bound on $\mathcal{L}_{N_t}(s)$ from (5) is obtained by using Jensen's inequality:

$$\begin{aligned} \mathcal{L}_{N_t}(s) \ &\overset{(a)}{\geq} \ \exp\left[-\frac{1-p}{p}(1-e^{-s})\int_{\mathbb{R}^2}\mathbb{E}_\nu\mathbf{1}(o\to x,\nu)\mathrm{d}x\right] \\ &\overset{(b)}{=} \ \exp\left[-\frac{1-p}{p\beta^2}(1-e^{-s})\right] \end{aligned}$$

where (a) follows from Jensen's inequality and (b) follows since $\mathbb{E}_\nu\mathbf{1}(o \to x,\nu) = \exp(-\lambda\beta\pi\|x\|^2)$. This is the Laplace transform of a Poisson random variable with mean $\frac{1-p}{p\beta^2}$ which implies the following lower bound on the probability of a transmit node being isolated:

$$\mathbb{P}(N_t = 0) \;\; \geq \;\; \exp\left(-\frac{1-p}{p\beta^2}\right)$$

We next evaluate the in-degree distribution of a receive node. Since the point process is stationary, the distribution of $N_r(x)$ is the same for all receivers x.

Proposition 3: The average in-degree $\mathbb{E}[N_r(x)]$ of a node in g is β^{-2}. When $\beta > 1$, N_r is distributed as a Bernoulli random variable with mean β^{-2}.

Proof: We have $N_r(x) \stackrel{d}{=} N_r(o)$ and hence,

$$\begin{aligned}
\mathbb{E}[N_r(o)] &= \mathbb{E}\left[\sum_{y\in\phi} \mathbf{1}_{\phi_t}(y)\mathbf{1}(y \to o, \phi_t)\right] \\
&= \lambda p \int_{\mathbb{R}^2} \mathbb{E}_{\phi_t}\left[\mathbf{1}(y \to o, \phi_t)\right] \mathrm{d}y \\
&= \lambda p \int_{\mathbb{R}^2} \exp\left(-\lambda p \pi \beta^2 \|y\|\right) \mathrm{d}y \\
&= \beta^{-2}.
\end{aligned}$$

If $\beta > 1$, at most one transmitter can connect to any receiver, so N_r is Bernoulli. Since $\mathbb{E}\left[N_r(x)\right] = \beta^{-2}$, we have $N_r(x) \sim$ Bernoulli(β^{-2}). ∎

Observe that the in-degree $N_r(x)$ does not depend on p. This is because of the homogeneity of the protocol model and the point process. Also observe that $\mathbb{E}[N_t(x)]$ and $\mathbb{E}[N_r(x)]$ are spatial averages and not time averages.

In the next subsection we characterize the length of the edges formed in the graph g.

B. Edge length distribution of the graph g

Each transmitter can potentially talk to many receivers. In this subsection we will analyze the length of edges formed by a typical transmitter. Since the process is stationary, we can just consider a typical transmitter at the origin and find its edge length distribution.

Definition 2: We define the average length over which a transmitter at the origin can communicate as

$$L = \frac{\mathbb{E}\left[\sum_{x\in\phi_r} \|x\| \mathbf{1}(o \to x, \phi_t)\right]}{\mathbb{E}\left[\sum_{x\in\phi_r} \mathbf{1}(o \to x, \phi_t)\right]}. \tag{9}$$

Observe that we are taking the average of the numerator and the denominator separately.

Proposition 4: The average length over which a transmitter can connect is given by

$$L = \frac{1}{2\beta\sqrt{\lambda p}}$$

Proof: The probability that a transmitter located at origin can connect to a receiver located at x, is given by

$$\mathbb{E}\mathbf{1}(o \to x, \phi_t) = \exp\left(-\lambda p \pi \beta^2 \|x\|^2\right) \tag{10}$$

By the Campbell-Mecke theorem [13], the numerator of L is

$$\lambda(1-p)\int_{\mathbb{R}^2} \|x\| \exp\left(-\lambda p \pi \beta^2 \|x\|^2\right) \mathrm{d}x,$$

and the denominator is given by $\mathbb{E}[N_t(o)]$ from which the result follows. ∎

We observe that the average length L is equal to $1/\beta$ times the mean distance to the nearest neighboring transmitter. This is intuitive since the range of transmission is limited by the nearest interferer. Let x_{max} denote the maximum distance that a transmitter at the origin can connect to. We now provide bounds on the CDF of x_{max}.

Proposition 5: The CDF of the maximum distance that a transmitter can connect to is bounded by

$$\exp\left(-\frac{(1-p)}{p}\beta^{-2}e^{-\lambda p \pi \beta^2 y^2}\right) \leq F_{x_{max}}(y) \quad \leq \quad 1 - \frac{\exp\left(-p\lambda\pi(1+\beta)^2 y^2\right)}{1+\frac{p}{1-p}(1+\beta)^2}.$$

Proof: The CDF is given by

$$\begin{aligned}
\mathbb{P}\left(\max_{x\in\phi_r}\{\|x\|\mathbf{1}(o\to x,\phi_t)\}\leq y\right) &= \mathbb{E}\left[\prod_{x\in\phi_r}\mathbf{1}\left(\|x\|\mathbf{1}(o\to x,\phi_t)\leq y\right)\right] \\
&\stackrel{(a)}{=} \mathbb{E}\left[\exp\left(-(1-p)\lambda\int_{\mathbb{R}^2}\mathbf{1}\left(\|x\|\mathbf{1}(o\to x,\phi_t)>y\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(-(1-p)\lambda\int_{B(o,y)^c}\mathbf{1}(o\to x,\phi_t)\mathrm{d}x\right)\right]
\end{aligned}$$

where (a) follows from the probability generating functional of the Poisson point process. A lower bound can now be obtained by using Jensen's inequality. We have

$$\begin{aligned} F_{x_{max}}(y) &\geq \exp\left(-(1-p)\lambda \int_{B(o,y)^c} \exp\left(-\lambda p \pi \beta^2 \|x\|^2\right) \mathrm{d}x\right) \\ &= \exp\left(-\frac{(1-p)}{p}\beta^{-2} e^{-\lambda p \pi \beta^2 y^2}\right). \end{aligned}$$

An upper bound can be obtained by proceeding as in (8):

$$F_{x_{max}}(y) \leq \mathbb{E}\left[\exp\left(-(1-p)\lambda \int_{B(o,y)^c} \rho(x,\phi_t)\mathrm{d}x\right)\right]$$

$$\begin{aligned} = &\int_0^{y(\beta+1)} \exp\left(-\lambda p \pi a^2\right) 2p\pi\lambda x \mathrm{d}x \\ &+ \int_{y(\beta+1)}^{\infty} \exp\left(-(1-p)\lambda\pi\left(\frac{x^2}{(\beta+1)^2} - y^2\right)\right) \exp\left(-\lambda p \pi x^2\right) 2p\pi\lambda x \mathrm{d}x \end{aligned}$$

$$\begin{aligned} &= 1 - \exp\left(-\lambda p \pi y^2 (1+\beta)^2\right) + \frac{\exp\left(-\lambda\pi p(\beta+1)^2 y^2\right)}{1 + \frac{1-p}{p}(\beta+1)^{-2}} \\ &= 1 - \frac{\exp\left(-\lambda\pi p(\beta+1)^2 y^2\right)}{1 + \frac{p}{1-p}(\beta+1)^2} \end{aligned}$$

■

IV. The Time Evolution Graph $G(0,n)$

In the previous section we analyzed the connectivity graph formed at a particular time instant. In this section we will consider the superposition of these graphs and study how the connectivity evolves over time.

A. Mean node degree in $G(0,n)$

Average node degree: Since each node is a transmitter with probability p or a receiver with probability $1-p$, the average (averaged over time and space) node degree in $G(0,n)$ is given by

$$np\mathbb{E}\left[N_t\right] = n(1-p)\mathbb{E}\left[N_r\right] = n(1-p)\beta^{-2}.$$

We will use $N_t(x,k)$ to denote the out-degree of node x at time instant k (similarly define $N_r(x,k)$).

Temporal correlation: The node degrees $N_t(x,k_1)$ and $N_t(x,k_2)$ (similarly with $N_r(x,k)$) are not independent for different time slots k_1 and k_2. This is because of the underlying point process ϕ which is common to both the random variables. For example consider the probability that a transmitter located at the origin is isolated in $g(1)$ and $g(2)$.

$$
\begin{aligned}
& \mathbb{P}\left(N_t(o,1)=0, N_t(o,2)=0\right) \\
= \;& \mathbb{E}\left[\prod_{x\in\phi}\left(1-\mathbf{1}_{\phi_r(1)}(x)\mathbf{1}(o\to x,\phi_t(1))\right)\left(1-\mathbf{1}_{\phi_r(2)}(x)\mathbf{1}(o\to x,\phi_t(2))\right)\right]
\end{aligned}
$$

which is of the form $\mathbb{E}\prod_{x\in\phi} f(x,\phi)$. Such an expression can be analyzed as an infinite series using the approach in [14]. These correlations make the analysis of the graph $G(0,n)$ difficult. By assumption (2) we neglect these temporal correlations.

1) Average number of paths between o and x by time n: Add two points to ϕ, one at the origin o and another at location x. Let $A(k), 1\leq k\leq n$ denote the adjacency matrix of the graph $g(k)$, with the nodes ordered with respect to the distance from origin. Then $A_{ij}(k)=\{\mathbf{1}_{E_k}((x_i,x_j))\}$. We observe that $A_{ij}(k), 1\leq k\leq n$, is iid over k for all i,j.

Lemma 1: With the assumption (2), the average number of paths (need not be edge-disjoint) between o and x is given by

$$
\mathbb{E}[N_n(x)] = \sum_{k=1}^{n}\binom{n}{k}\frac{p(1-p)}{k}\left(\frac{(1-p)}{\beta^2}\right)^{k-1}\exp(-\frac{p}{k}\lambda\pi\|x\|^2\beta^2) \tag{11}
$$

Proof: Let $\mathcal{A}^n=\prod_{i=1}^{n}(I+A(i))$. So the total number of edges $N_n(x)$ will be the entry in the row corresponding to the origin and the column of x in $\mathcal{A}^n$. $N_n(x)=\sum_{k=1}^{n}\tilde{N}_n^k(x)$ where $\tilde{N}_n^k(x)$ denote the number paths of length k between o and x by time n. We first evaluate $\mathbb{E}[\tilde{N}_n^k(x)]$. From the matrix $\mathcal{A}^n$ and the iid property of $A(k)$, we have

$$
\mathbb{E}[\tilde{N}_n^k(x)] \;=\; \binom{n}{k}\mathbb{E}\left[\sum^{\neq}_{x_1\ldots x_{k-1}\in\phi\setminus\{o,x\}}\prod_{m=0}^{k-1}\mathbf{1}_{E_m}((x_m,x_{m+1}))\right]
$$

where $x_0 = o$ and $x_k = x$ and $\sum^{\neq}$ denotes summation over disjoint point sets. So we are not counting loops. Using (2) and the Campbell-Mecke theorem we have

$$\begin{aligned}\mathbb{E}[\tilde{N}_n^k(x)] &= \binom{n}{k}\lambda^{k-1}\int\prod_{m=0}^{k-1}\mathbb{P}((x_m,x_{m+1})\in E_1)\mathrm{d}x_1\ldots\mathrm{d}x_{k-1}\\ &= \binom{n}{k}\lambda^{k-1}(p(1-p))^k\cdot\\ &\quad\int\exp\left(-\lambda p\pi\beta^2\sum_{i=0}^{k}\|x_i-x_{i+1}\|^2\right)\mathrm{d}x_1\ldots\mathrm{d}x_{k-1},\end{aligned}$$

which, after some manipulation, yields (11). ■

We have the following observations regarding the average number of paths between o and x in n time instants:

1) For large $\|x\|$ the main contributors are $\mathbb{E}[\tilde{N}_n^k(x)]$, for k large, i.e., multihop routing becomes important as $\|x\|$ increases.
2) One can think of $\mathbb{E}[N_n(x)]$ as the average *path diversity* that is offered to a packet.

B. *Asymptotic analysis of* $G(0,n)$

We first define the connection time between two nodes. For $x, y \in \phi$, we denote the *path formation time* between x and y as

$$T(x,y) = \min\{k\ :\ G(0,k) \text{ has a path from } x \text{ to } y\}.$$

For general $x, y \in \mathbb{R}^2$, define $T(x,y) = T(x^*, y^*)$ where x^* (resp. y^*) is the point in ϕ closest to x (resp. y), with some fixed deterministic rule for breaking ties (no ties almost surely). Since the point process is isotropic, it is sufficient for most cases to consider destinations along a given direction. For notational convenience we define for $y \in \mathbb{R}$, $T(x,y) = T(x,(y,0))$.

This path formation time is the minimum time required for a packet to propagate from a source x to its destination y in an ALOHA network. In this section we show that this propagation delay increases linearly with the source-destination distance. Similar to $T(x,y)$ we define

$$T_n(x,y) = \min_{k>n}\{k-n\ :\ G(n,k) \text{ has a path from } x \text{ to } y\}.$$

Let

$$\tilde{B}_t = \left\{x \colon x \in \mathbb{R}^2,\ T(o,x) \le t\right\}$$

denote the set of points which can be reached from the origin by time t. We denote the farthest distance reached by time t as

$$D_t = \sup\left\{\|x\| \colon x \in \mathbb{R}^2,\ T(o,x) \le t\right\}.$$

The evolution of the graph $G(0,n)$ is similar to the growth of an epidemic on the plane and one can relate this problem to the theory of Markovian contact processes [7] which was used to analyze the growth of epidemics. We now provide bounds on the path formation time between two points.

Direct connection: By assumption (2), we have that the time taken for a direct connection between two points x and y is a geometric random variable with parameter

$$\eta(x,y) = p(1-p)\mathbb{E}\left[\exp\left(-\lambda p\pi\beta^2\|x^* - y^*\|^2\right)\right]$$

where the average is with respect to the distribution of $\|x^* - y^*\|$. *For most of the analysis we assume $\|x-y\|$ to be large so that $\|x^* - y^*\| \approx \|x-y\|$. Henceforth we shall not distinguish between x and x^* .*

Lemma 2: For large $x \in \mathbb{Z}^+$, the tail probability of $T(o,x)$ is bounded as

$$\mathbb{P}(T(o,x) > k) \le I_{1-\eta(o,a)}(k+1,m)$$

for any $1/\sqrt{\lambda} < a < x$, where $m = \lceil x/a \rceil$ and

$$I_{1-\eta(o,a)}(k+1,m) = \frac{(m+k+1)!}{m!(k+1)!}\int_0^{\eta(o,a)} t^m(1-t)^k dt$$

is the regularized beta function.

Proof: We imposed $1/\sqrt{\lambda} < a$ so that $\|(0,a)^*\| \approx a$. Let $t_1(a)$ be the time for an edge to form between o and $(a,0)$ and $t_2(a)$ be the time required for a direct connection to form between $(a,0)$ and $(2a,0)$ after the first edge is formed. Similarly define $t_k(a)$ to be the time required for a connection to form between $((k-1)a,0)$ and $(ka,0)$ after all the previous $k-1$ connections are formed. See Figure 3. By assumption (2), we have $t_i, 1 \le i \le m$, to be independent. So we

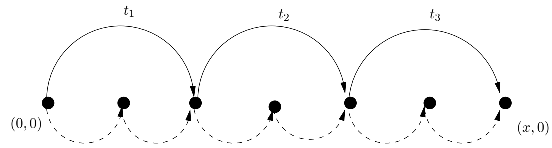

Figure 3. The node at the origin can transfer packets to a node at $(x, 0)$ by using the shorter hops (indicated by dashed line) or using longer hops (solid lines). Longer hops are difficult to form but only few are required to reach the destination. Shorter hops are easy to form but a higher number is required to reach the destination.

have

$$T(o,x) \leq \sum_{i=1}^{m} t_i(a) \tag{12}$$

The t_i are iid geometrically distributed with parameter $\eta(o,a)$. Hence we have

$$\begin{aligned} \mathbb{P}(T(o,x) > k) &\leq \mathbb{P}\left(\sum_{i=1}^{m} t_i(a) > k\right) \\ &\overset{(a)}{=} I_{1-\eta(o,a)}(k+1, m), \end{aligned} \tag{13}$$

where (a) follows from the fact that the sum of geometric random variables follows a negative binomial distribution. ■

In the following arguments we rely on the spatial subadditivity of $T(o,x)$ to analyze the asymptotic properties. Subadditivity of random variables is a powerful tool which is often used to prove results in percolation and geometric graph theory. The problem of finding the minimum delay path is similar to the problem of first-passage percolation. From the definition of $T(o,y)$, we observe that

$$T(o,y) \leq T(o,x) + T_{T(o,x)}(x,y). \tag{14}$$

We also have that $T_{T(o,n)}(x,y) \overset{d}{=} T(x,y)$ from the way the graph process is defined. Observe that (14) resembles the triangle inequality (specially if $T_{T(o,y)}(x,y)$ was $T(x,y)$) and thus provides a pseudo-metric, which holds in FPP problems and is the reason that the shortest paths in FPP are called geodesics. In the next two lemmata we show that the average time for a path to form between two nodes scales linearly with the distance between them.

Lemma 3: The time constant defined by

$$\mu = \lim_{x\to\infty} \frac{\mathbb{E}T(o,x)}{x}$$

exists when $x \in \mathbb{Z}^+$.

Proof: Let $y \in \mathbb{Z}^+$. From (14), we have

$$T(o, y+x) \leq T(o,y) + T_{T(o,y)}(y, y+x). \tag{15}$$

From the definition of the graph, E_k does not depend on $E_i,\ i < k$. Hence we have that $T_{T(o,y)}(y, y+x)$ has the same distribution as $T(y, y+x)$. Also from the invariance of the point process ϕ, we have $T(y, y+x) \overset{d}{=} T(o,x)$. Taking expectations of (15), we obtain

$$\mathbb{E}T(o, y+x) \leq \mathbb{E}T(o,y) + \mathbb{E}T(o,x),$$

and the result follows from the basic properties of subadditive sequences. ∎

We do not require assumption (2) to prove Lemma 3. Consistent with the FPP terminology we will call μ the time constant of the process. We now prove that the time constant for the modified protocol model is always greater than zero and finite.

Lemma 4: For the modified protocol model

$$\frac{\beta\sqrt{p\pi\lambda}}{\sqrt{\ln(1+p(1-p))}} \leq \mu \leq \frac{\beta\sqrt{2\pi\lambda}\exp(1/2)}{(1-p)\sqrt{p}} \tag{16}$$

Proof: Upper bound: Taking expectation on both sides of (12) and since $t_i(a)$ are identically distributed for all i (we drop the i in the subscript for notational convenience), we have

$$\begin{aligned} \mathbb{E}T(o,x) &\leq \left\lceil \frac{x}{a} \right\rceil \mathbb{E}t(a) \\ &\leq \left(\frac{x}{a} + 1 \right) \mathbb{E}t(a). \end{aligned}$$

Dividing both sides by x and taking the limit we obtain

$$\lim_{x \to \infty} \frac{\mathbb{E}T(o,x)}{x} \leq \frac{\mathbb{E}t(a)}{a}.$$

Assuming $\|(0,a)^*\| \approx a$, $t(a)$ is a geometric random variable with mean $p(1-p)\exp(-p\lambda\pi\beta^2a^2)$. So we get

$$\mu \leq \frac{\exp(p\lambda\pi\beta^2a^2)}{ap(1-p)}.$$

The upper bound is obtained by using $a = 1/(\beta\sqrt{2p\lambda\pi})$ for which the right hand side of the above equation is minimized.

Lower bound: Taking large hops to reach the destination requires fewer hops but the success probability for each hop would be small and hence it takes more time to connect. On the other hand taking smaller hops will result in a higher probability of success for each hop and result in a smaller time of connection, but we require a large number of hops to get to the destination. We will use the tradeoff between the hopping distance versus time to show that

$$\mathbb{P}(T(o,x) < cx) \to 0$$

as $x \to \infty$ for some positive c and $x \in \mathbb{Z}^+$. This implies $\mathbb{E}T(o,x)/x > c$ for some $c > 0$ and hence $\mu > 0$. For the sake of notational convenience let cx be identified with $\lceil cx \rceil$. So to evaluate the event $\{T(o,x) \leq cx\}$, we consider only those paths which have a maximum of cx hops. By the union bound we have

$$\mathbb{P}(T(o,x) < cx) \leq \sum_{i=1}^{cx} p_i \tag{17}$$

where $p_i = \mathbb{P}(T(o,x) < cx \mid \text{there is a path from } o \text{ to } x \text{ with } i \text{ hops})$. The time to form any single direct link between two nodes o and y is a geometric random variable with parameter $\eta(o,y) = p(1-p)\exp(-c_1\|o-y\|^2)$, where $c_1 = \lambda p \pi \beta^2$. So the times to form the hops in a k-hop path between $o, x_1, x_2 \ldots, x_{k-1}, x$ are a series of geometric random variables t_i with parameters $\eta(x_{i-1}, x_i)$ which are independent because they occur in different time slots, see (2). Let $\xi > 0$. So we have

$$p_k \leq \mathbb{P}(\sum_{i=1}^{k} t_i < cx).$$

We also have that if $t_1, ..., t_i$ are independent geometric random variables with parameters p_i, then

$$\mathbb{P}(\sum_{i=1}^{k} t_i < a) \leq \exp(\xi a) \left(\frac{e^{-\xi}}{1 - e^{-\xi}} \right)^k \prod_{i=1}^{k} p_i \tag{18}$$

for any $\xi > 0$ (follows from Chernoff bound). So

$$\begin{aligned} p_k &\stackrel{(a)}{\leq} \exp(\xi cx) \frac{1}{(\exp(\xi)-1)^k} \prod_{i=1}^{k} \eta(x_{i-1}, x_i) \\ &= \exp(\xi cx) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp(-c_1(\|o - x_1\|^2 \\ &\quad + \|x_2 - x_1\|^2 + \ldots + \|x_{k-1} - x\|^2)) \\ &\stackrel{(b)}{\leq} \exp(\xi cx) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp\left(-c_1 \frac{x^2}{k} \right). \end{aligned}$$

(a) follows from (18) and (b) follows from the fact that the minimum value of $\|x_1\|^2 + \|x_2 - x_1\|^2 + \ldots + \|x_{k-1} - x\|^2$ is x^2/k. So from (17), we have

$$\begin{aligned} \mathbb{P}(T(o,x) < cx) &\leq \sum_{k=1}^{cx} \exp(\xi cx) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp\left(-c_1 \frac{x^2}{k} \right) \\ &\stackrel{(a)}{\leq} cx \exp(\xi' cx) \exp\left(-\frac{c_1}{c} x \right), \end{aligned}$$

where (a) follows by choosing $\xi = \xi'$ such that $p(1-p)/(\exp(\xi')-1) < 1$ and using $k = cx$ for all the terms. The right hand side goes to 0 if $c < \sqrt{c_1/\xi'}$. Hence we have $\mathbb{E}[T(o,x)/x] > c$ which implies $\mu > c$. We can choose $\xi' = (1+\epsilon)\ln(1+p(1-p))$ for any $1 > \epsilon > 0$ and we then have the lower bound $c \geq (1-\epsilon)\sqrt{\frac{\lambda p \pi \beta^2}{(1+\epsilon)\ln(1+p(1-p))}}$. ■

In the modified protocol model we are considering, we do not have any power constraint. So any node can potentially connect to any receiver no matter how far it is but the probability decreases exponentially with distance and hence $\mu < \infty$. This is in contrast to standard first-passage percolation on a lattice where the probability distribution (CDF) on each edge should have a mass less than P_c at zero for $\mu < \infty$, where P_c is the bond percolation threshold of the lattice. If we had considered a power constraint, for example by putting a hard limit on the maximum link distance, $\|x - y\| < R$ (original protocol model), then there is no guarantee that the time constant $\mu < \infty$. We conjecture that if R is chosen so that the disk graph formed by placing disks of radius R around each node of ϕ percolates, i.e., for $R > \sqrt{1.435/\lambda}$ [15] then $\mu < \infty$. In deriving the lower bound we have used assumption (2). In practice the constants may change but the scaling with respect to the different parameters would remain the same. From the lower bound on μ we have that $\mu > 0$ when $p \to 0$, but as noted previously, this is an observation that is of mathematical interest only, since the noise-free assumption does not hold when $p \to 0$. We also observe that the lower bound on the time constant increases with p. From the upper and lower bounds we observe that μ scales like $\beta\sqrt{\lambda}$.

Since we do not have $T(o, x+y) \leq T(o, x) + T(o, y)$, Kingman's subadditive ergodic theorem [9] cannot be directly applied to (14). But since $T_{T(o,x)}(x, y) \stackrel{d}{=} T(x, y)$, there is hope that such a result holds. In the next lemma, we prove that this is indeed the case.

Lemma 5: Let μ be the time constant of the process,

$$\frac{T(o,x)}{x} \longrightarrow \mu, \quad x \to \infty \tag{19}$$

$x \in \mathbb{Z}^+$ and where the convergence is in L^2 and hence in probability.

Proof: From (14), and $T_{T(o,x)}(x, x+y) \stackrel{d}{=} T(o, y)$ and the fact that $T_{T(o,y)}(y, x+y)$ is independent of $T(o, y)$ (because of assumption (2)), we have

$$F_{x+y}(\xi) \geq (F_x * F_y)(\xi),$$

where F_x is the CDF of $T(o, x)$. $\mathbb{E}(T(o, x)^2) < \infty$ follows from Lemma 2. So we have a superconvolutive sequence and hence by Kesten's lemma [16], [17], [18, p. 120] holds[1]. ■

This result shows that with high probability, the delay required for a packet propagation scales linearly with distance.

Next we address the maximum distance that a packet travels by time n.

Lemma 6: The velocity of information propagation converges, i.e.,

$$\frac{D_n}{n} \to \mu_1 \tag{20}$$

$n \in \mathbb{Z}^+$, with probability one. Also $\mu_1 = \frac{1}{\mu} \in (0, \infty)$.

Proof: We have

$$D_{m+n} \geq D_m + \tilde{D}_{mn} \tag{21}$$

where $\tilde{D}_{mn}$ is the farthest distance reached in n time steps starting from the point which achieved D_m. We observe that $\tilde{D}_{mn} \stackrel{d}{=} D_n$. Hence from (21), we have that D_n forms a subconvolutive sequence and hence $-D_n$ forms a superconvolutive sequence. So if we show $\mathbb{E}[D_1^2] < \infty$, L^2

[1]To prove the a.e. convergence using Kesten's lemma, we would require that $T(o, n)$ be a monotone sequence, which is not true in our case.

convergence follows, by Kesten's lemma. We have

$$\mathbb{P}(D_1 < x) = \mathbb{P}(\text{The origin is not able to connect} \tag{22}$$

$$\text{to any node at distance greater than } x)$$

$$\stackrel{(a)}{>} \exp\left(-\frac{1-p}{p}\beta^{-2}e^{-\lambda p\pi\beta^2 x^2}\right) \tag{23}$$

where (a) follows from Proposition 5. From (23), we observe that the tail probability of D_1 decays exponentially fast with x and hence D_1 has a finite second moment. Since the sequence D_n is also monotonic, we have convergence with probability one. We also have $\mathbb{E}[D_n]/n > \mathbb{E}[D_1] > 0$. Using Lemma 7, one can also deduce $\mu_1 = 1/\mu$ and from Lemma 4, we have $\mu_1 < \infty$. ■

Next we prove the convergence of $\tilde{B}_t/t$ to a fixed set $B(o, \mu^{-1})$ (shape theorem). In (19), we considered the time required for connection between o and x along the e_1 direction. By the isotropy of the point process, and the process, we have that for any $e_\theta \in \{y \in \mathbb{R}^2 \colon \|y\| = 1\}$, $T(o, xe_\theta)/x \to \mu,\ x \in \mathbb{Z}^+$, in L^2. Until now we have shown the convergence of $T(o, xe_\theta)/x$ when x is an integer. In the next Lemma we will prove the same result when $x \in \mathbb{R}^2$. It follows closely the technique used in [19] to prove the shape theorem for Euclidean first-passage percolation.

Lemma 7: We have[2] for $x \in \mathbb{R}^2$, $\lim_{\|x\| \to \infty} T(o, x)/\|x\| \to \mu$, in L^2.

Proof: See the Appendix. ■

Theorem 1: (Shape theorem) When $\mu > 0$, $\tilde{B}_t/t$ converges in probability to $B(o, \mu^{-1})$. More precisely, $\forall \delta > 0,\ \epsilon > 0,\ \exists t_0$ such that

$$\mathbb{P}(B(o, (\mu^{-1} - \epsilon)) \subset \frac{\tilde{B}_t}{t} \subset B(o, (\mu^{-1} + \epsilon))) > 1 - \delta, \quad \forall t > t_0.$$

Proof: Follows from Lemma 7. The proof is similar to [19, p. 10]. ■

V. Simulation Results

In this section we illustrate the results using simulation results. For the purpose of simulation we consider a PPP of unit density in the square $[-50, 50]^2$. For most of the simulations, we use

[2] Note that we have replaced the integer index with a continuous variable over all directions.

$\beta = 1.2$, and we average over 200 independent realizations of the point process. In Figure 4, $\mathbb{E}T(o, x)$ is plotted with respect to x for different values of p. The time constant μ is plotted as a function of p in Figure 5. We make the following observations:

1) The time constant increases with the ALOHA parameter p.
2) In Figure 4, we observe that $\mathbb{E}T(o, x) \approx \mu(p)x + C(p)$, where $C(p)$ is a decreasing function of p and $\mu(p)$ is increasing. For smaller values of p, the time taken for a node to become a transmitter is large, but the probability of a successful transmission is also high because of the low density of transmitters. This results in a large $C(p)$ and smaller $\mu(p)$ for small p.
3) Figure 4 also implies that the presence of interfering transmitters causes the delay to increase when the packet has to be transmitted over longer distances. So when the packet transmission distance is large, it is beneficial to decrease the density of contending transmitters.
4) For each x, there is an optimal p which minimizes the delay, and the optimum p is a decreasing function of x.

For two nodes located at o and x and $\|x\|$ large, there will in general be many paths between o and x which form by time $\mu\|x\|$. From such an ensemble of delay-optimal paths, we will consider paths which have the minimum number of hops and call them *fastest paths*. In Figure 6, we show the average hop in these paths. We observe that for a given p, the average hop length decreases as the source-destination distance x increases. This shows that for larger source-destination distance, it is beneficial to use shorter hops since they are more reliable and form faster than longer hops. Also from Figure 5, we observe that for larger x, it is beneficial to be less aggressive in terms of spatial reuse and use a smaller p.

VI. Conclusions

Connectivity in a wireless network is dynamic and directed because of the MAC scheduler and the half-duplex radios. Since these properties are not captured in static graph models that are usually used, we have introduced a dynamic connectivity graph and analyzed its properties for ALOHA. We have shown that the time taken for a causal path to form between a source and a destination on this dynamic ALOHA graph scales linearly with the source-destination distance and have derived bounds on the pre-constants. This implies that every node can be reached in a

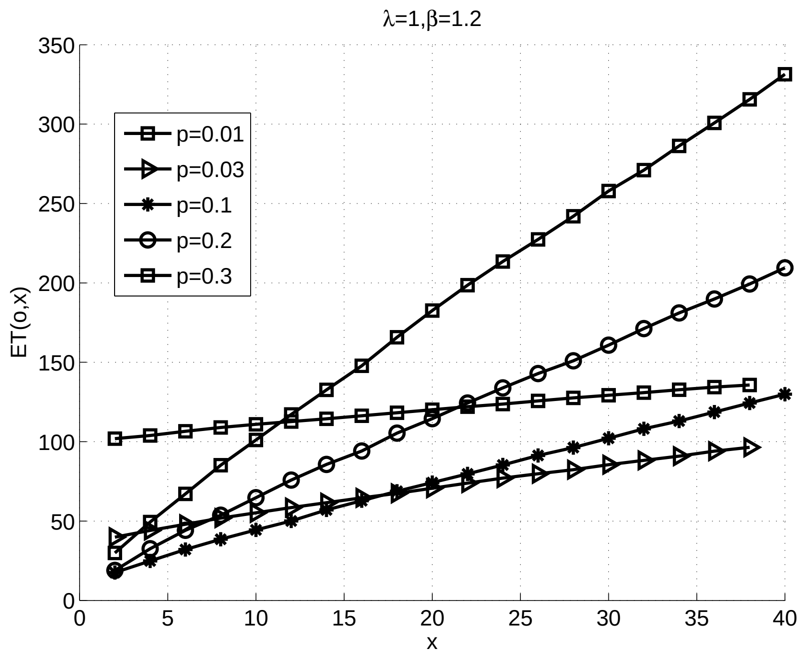

Figure 4. $\mathbb{E}T(o,x)$ as a function of x, for $\beta = 1.2$. We first observe the linear scaling of $\mathbb{E}T(o,x)$ with the distance x and that the slope increases with p. Also for small values of x we observe that $\mathbb{E}T(o,x) \approx p^{-1}$ since for small x the path delay time is dominated by the MAC contention time. For small values of p, once the source is a transmitter, long edges form due to the low interference.

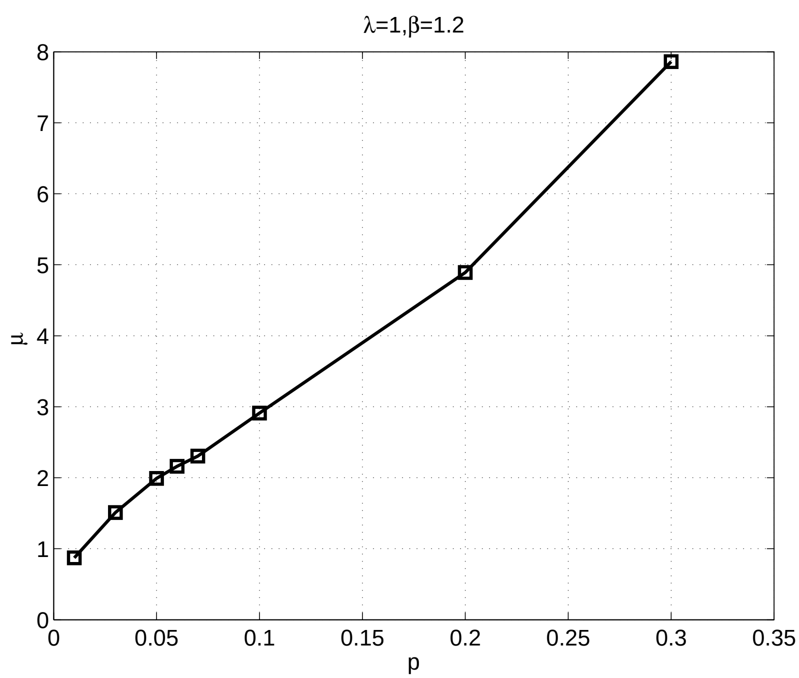

Figure 5. The time constant μ as a function of p, for $\beta = 1.2$

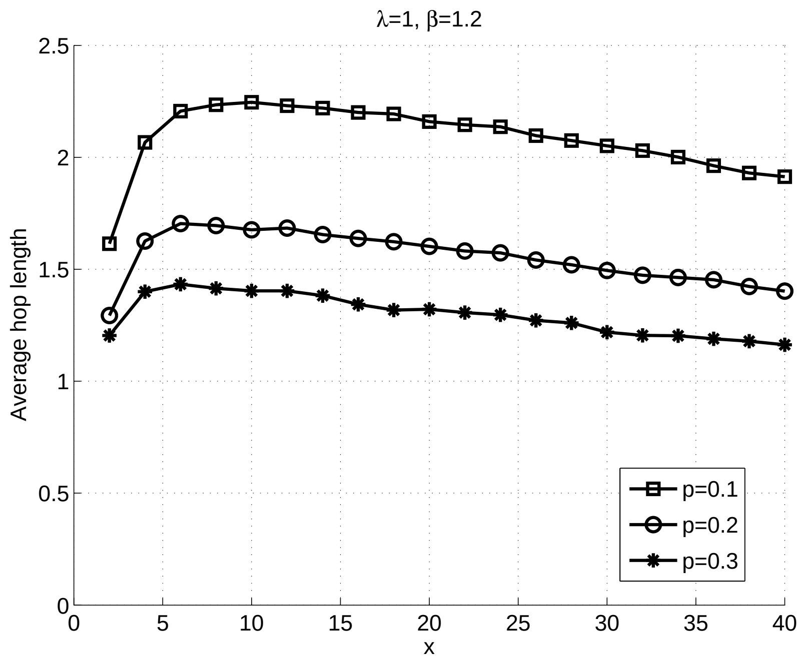

Figure 6. Average hop length in the fastest path versus the source-destination distance.

time that is linear with the distance. The result shows that one does not require full connectivity in a single instant; hence the requirement of a giant connected component (percolation) in a network with interference [10] is greatly relaxed. Hence, e.g., in a route discovery flooding algorithm, the time to find the route scales linearly with the diameter of the network. We also showed that the nodes which are able to connect to a node at the origin by a certain time are located in a circle whose radius that scales linearly with time (shape theorem). By simulations we showed that it is beneficial to use higher value of the ALOHA contention parameter for smaller source-destination distances and lower value for large distances, and that the average hop length of the fastest paths first increases rapidly but then decreases slowly as a function of the source-destination distance. This observation provides some insight how to choose the hop length for efficient routing in ad hoc networks.

APPENDIX

In this Appendix we prove Lemma 7.

Proof: For a random variable X let $|X|_2$ denote the L_2 norm $\sqrt{\mathbb{E}(X^2)}$. We follow the idea of the proof of Lemma 4 in [19]. Fix $\epsilon > 0$. Pick the vectors $u_i \in \{y \in \mathbb{R}^2 : \|y\| = 1\}, i \in \{1, .., m\}$, such that

$$R = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m} B(iu_j, \epsilon i)$$

So R covers most of the region of $\mathbb{R}^2$ except some region near origin. For any x, choose $i = i(x), j = j(x)$ such that $x \in B(iu_j, \epsilon i)$. Observe that $i \to \infty$ as $\|x\| \to \infty$. We also have $\|x - iu_j\| < \epsilon i$. From Lemma 5 we know that for large i, $|T(o, iu_j) - \mu i|_2 \leq \epsilon i$. So we get

$$\begin{aligned} |T(o,x) - \mu\|x\||_2 &\leq |\mu i - \mu\|x\||_2 + |T(o,iu_j) - \mu i|_2 + |T(o,x) - T(o,iu_j)|_2 \\ &\leq \mu\epsilon i + \epsilon i + |T(o,x) - T(o,iu_j)|_2 \end{aligned} \tag{24}$$

We also have from (15),

$$\begin{aligned} |T(o,x) - T(o,iu_j)|_2 &\leq |T_{T(o,x)}(x, iu_j)|_2 \\ &\overset{(a)}{=} |T(o, \|x - iu_j\|)|_2 \end{aligned}$$

where (a) follows from the stationarity of ϕ and the fact that $T_{T(o,x)}(x, iu_j) \overset{d}{=} T(o, \|x - iu_j\|)$. Using the notation of Lemma 2, we obtain

$$T(o, \|x - iu_j\|) \leq \sum_{i=1}^{\lfloor \|x - iu_j\| \rfloor} t_i + t_{\lfloor \|x - iu_j\| \rfloor}^{\|x - iu_j\|}$$

where $t_{\lfloor \|x - iu_j\| \rfloor}^{\|x - iu_j\|}$ is the time for a direct connection between $\lfloor \|x - iu_j\| \rfloor$ and $\|x - iu_j\|$. Also t_i are identical and $c_4 = |t_i|_2 < \infty$. Using triangle inequality we have

$$\begin{aligned} |T(o, \|x - iu_j\|)|_2 &\leq \sum_{i=1}^{\lfloor \|x - iu_j\| \rfloor} |t_i|_2 + |t_{\lfloor \|x - iu_j\| \rfloor}^{\|x - iu_j\|}|_2 \\ &= \lfloor \|x - iu_j\| \rfloor\, c_4 + |t_{\lfloor \|x - iu_j\| \rfloor}^{\|x - iu_j\|}|_2 \\ &= c_5 \lfloor \|x - iu_j\| \rfloor \\ &\leq c_5 \epsilon i \end{aligned}$$

Substituting in (24), we have

$$\begin{aligned} |T(o,x) - \mu\|x\||_2 &\leq \mu\epsilon i + \epsilon i + c_5 \epsilon i \\ &= i\epsilon(\mu + 1 + c_5) \end{aligned}$$

Dividing both sides by $i = i(x)$ we have $|T(o,x) - \mu\|x\||_2/i(x) < \epsilon(\mu + 1 + c_5)$. Hence $|T(o,x) - \mu\|x\||_2/\|x\| \to 0$. ■

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COMMENTS ON REVISED VERSION

Note: Many thanks to the reviewers for their detailed comments, they have led to a substantial improvement of the quality of this paper. In the following, we are listing all the reviewers concerns (in italics) and interleave our reply.

Reviewer 1 Comments:

1) *I think the main goal of the paper is the computation of the delay to form a causal (directed) path between a pair of nodes in this network model. This a first passage percolation problem, which has not been solved in the protocol model with interferences, and which I find very interesting. Hence the second part of the paper (Sections IV. A. and V) seems to be the core of the paper. The first part is a list of properties of this class of random graphs. They are interesting on their own, but I fail to see where they help in solving the problem stated as the goal of the paper. The two parts appear therefore as quite unrelated. This is reflected by the notation n which means always a time in Sections II, III, IV 1 and then suddenly always a distance of a node from the origin from Section IV.A on. This confusion (In eq (18), n is a time, in (22) n is to be "interpreted as $ne1$ where $e1$ is the unit vector $(1,0)$") leads to errors: and expression (23) is not correct. The first part could therefore be put in appendix.*

In a wireless network the set of transmitting nodes changes almost at every time instant because of the MAC scheduler. The connectivity of a wireless network is generally analyzed as a static random geometric graph. But such a model to analyze the network would lead to more constraints on the system. For example the power required for percolation of the SINR graph would be much higher than for connectivity on the dynamic graph. In this paper, we consider an ad hoc network with half-duplex radios which uses multihop routing and slotted ALOHA for the network MAC contention. We introduce a random dynamic multi-digraph that models the connectivity of a wireless network more appropriately. The main motivation of the paper is to introduce the concept of dynamic graphs and study their properties for the ALOHA MAC protocol. The first part serves as a gentle introduction to the problem we are considering and leads to a natural continuity in understanding the final dynamic graph. We certainly agree that all the results are not used in the second part but we feel that the first part is of interest in its own right and helps the readers understand the limitations of analyzing a network using static graphs. We made the proofs of the first part more clear and easier to understand. The confusion about using n as time and space variable is taken care in this revision, and the space variables are changed to x and y.

2) *It is technically sound, even if some notations should be made more consistent (for example, sometimes you put ϕ_t explicitly and sometimes not, which does not add to clarity) or better defined (for example, $B(0,y)^c$ is not defined on page 8), if statements should be made rigorous (for example, the statement "$1(o \to x)$ does not change of x is scaled by...." at the end of p. 5 holds in distribution only) and should be made at the right place (you explain the notation $1(o \to x,)$ on p. 6 even though it has been used already much earlier).*

These mistakes are corrected in the revised version.

3) *The second part of the paper is the most interesting one, and the challenge here is the temporal correlations between links. Even though links are not independent, I do not see why you cannot use the usual sub-additive ergodic theorem (as exposed for example in Liggett's textbook and used for example in [8]). Essentially you do not have indeed that $T(0,n+m) \leq T(0,n) + T(n,n+m)$ but you have something very close: $T(0,n+m) \leq T(0,n) + T_s(n,n+m)$ where $s = T(0,n)$. Now, $T(n,n+m)$ and $T_s(n,n+m)$ are identical in distribution, and $T_s(n,n+m)$ is independent from $T(0,n)$ because of assumption (3), hence shouldn't we be able to use the theorem at least with assumption (3)?*

Kingman's subadditive theorem requires the stronger statement $T(0,n+m) \leq T(0,n) + T(n,n+m)$. Kingman's proof does not follow through with $T(n,m+n)$ replaced by another random variable $T_s(n,n+m)$ with identical distribution. A similar scenario occurs in the spatial spread of epidemics, for example [16]. In the spread of an epidemic, one can think of $T(m,n)$ as the time required for a disease to spread from a individual located at m to one located at n. Even in this case, Kestens lemma has to be used to show a similar convergence as in our work.

4) *Now, you mention in Lemma 3 that the limit μ exists and that $\mu \in [0, \infty)$ and next in Lemma 4 that $\mu > 0$. I agree with the result of Lemma 4 that that $\mu > 0$, but I do not see where you prove that $\mu < \infty$. Shouldn't we need λ to be large enough, to ensure the existence of infinite component in the connectivity graph? In general, the percolation of the underlying cumulative graph is the one that takes the most work to be established: unless I missed it somewhere, one cannot consider the result as proven without proving carefully the step $\mu < \infty$, even I think it must be true. I guess that with this proof, and the slight adaptation of this sub-additive lemma, all conditions in the Liggett's sub-additive theorem are verified, and thus the results hold.*

In the revised version we provide the following upper and lower bounds in Lemma 4:

$$\frac{\beta\sqrt{p\pi\lambda}}{\sqrt{\ln(1+p(1-p))}} \leq \mu \leq \frac{\beta\sqrt{2\pi\lambda}\exp(1/2)}{(1-p)\sqrt{p}} \tag{25}$$

In the modified protocol model we are considering, we do not have any power constraint. So any node can potentially connect to any receiver no matter how far it is but the probability decreases exponentially with distance and hence $\mu < \infty$. This is in contrast to standard first passage percolation on a lattice where the probability distribution (CDF) on each edge should have a mass less than P_c at zero for $\mu < \infty$, where P_c is the bond percolation constant of the lattice. If we had considered a power constraint, i.e., a transmitter x can connect to a receiver y if the original protocol model is satisfied i.e., $\|x-y\| < R$, then there is no guarantee that the time constant $\mu < \infty$. We conjecture that if R is chosen so that the disc graph formed by placing discs of radius R around each node of ϕ percolates, then $\mu < \infty$.

5) *Assumption (3) is central to the results, and it would be helpful to comment more on its practical validity.*

In essence assumption (3) ((2) in the revised version) implies that the interference encountered at each node is independent across time. This statement is correct if one conditions on the underlying point process ϕ since the ALOHA protocol chooses independent sets across time. In reality interference is not independent but almost because of the MAC protocol. We have included the comments in the revised version after the assumption.

6) *There are some other glitches however in that part of the paper, which need to be carefully proofread. For example I don't see how taking expectations in (27) yields the strict inequality of (28). I don't understand the term "conflict" in the first sentence of Proof of Lemma 4 ("The basic idea is to use the conflict that taking large hops etc"). The constant c is introduced twice after (29).*

The strict inequality is a typo and is corrected to "$\leq$". The "conflict" sentence is removed in the revised version. It has been replaced by the following argument: " Taking large hops to reach the destination requires fewer hops but the success probability for each hop would be small and hence it takes more time to connect. On the other hand taking smaller hops will result in higher probability of success for each hop and result in a smaller time of connection, but we require a large number of hops to get to the destination. We will use the tradeoff between the hopping distance versus time to show the lower bound on μ. "

7) *The discussion of the dependence of $T(0, n)$ and of μ on the probability p is very interesting (even if it can be made only by simulations). Is $\mu(p)$ a strictly increasing function for all values p, even those very small? Why? Can you prove it? What is $\lim \mu(p)$ for $p \to 0$? If p is large, I understand that interferences increase delay. But if p is very small, the picture is less clear to me. I understand that there is an optimal p for $T(0, n)$. It would be good to relate this discussion with the paper by "F. Baccelli et al, An aloha protocol for multihop mobile wireless networks, IEEE Transactions on Information Theory".*

From the upper and the lower bounds in (25), it *seems* that $\mu(p)$ is not a increasing function of p. But from simulations, we observe that $\mu(p)$ is increasing with p when p is not too close to zero. The problem occurs when p is close to zero. Then the probability that a node can become a transmitter is very small, but a node can transmitt to a large distance and it is not clear which dominates the other.

For the model we consider, we also observe that $\mu \downarrow \beta\sqrt{\pi\lambda}$ when $p \to 0$ from (25). But from a practical perspective, p close to zero would lead us from the interference-limited region into the noise-limited region. The modified protocol model we are considering does not take into account the noise and hence the results obtained when

p very close to zero may shift significantly from practice. In the paper by Baccelli, the emphasis is to maximize the distance transmitted and in our case the emphasis is to decrease the path formation time, and we do not see a direct connection between the two.

8) *The presentation of the paper should be improved. Some explanations of technical computations would be helpful, for example from the first to the second line on top of page 9; as well on top of page 11. (b) is not explained in set of equations at bottom of page 14. The proofs should also be made self-contained if possible (for example, remind the reader what a regularized beta function is in page 12): Typos are left, which can be easily caught by a spellchecker. The paper appears to be written quite hastily at some places (for example, there is a Section IV A on page 11, but not Section IV B, there is a Section IV 1) on page 10 but no Section IV 2) ?*

These problems are rectified in the revised version.

Reviewer 2 Comments:

1) *I find that the definitions, denotations, and some interpretations are not quite in line with existing results, in the context of communications and networking. For example, for the minimum delay to setup a path, this problem has been studied by reference [1] and succeeding works by considering link capacity, schedules, and interference. In this paper, the problem is formulated differently. The question is that the results are not consistent. Therefore, readers may be confused about the overall problem and findings. This does not mean that the authors need to follow others' methods, but it is important to point out similarities and differences because many readers may have read Kumar's paper and some of Dousse's papers on similar topics. The contributions of the papers are not very clear. If we take the main contribution as stated in the abstract, "the delay (to form a path) scales linearly with the distance" then it is hard to justify that this is a significant contribution. I would expect to see more understandings and comparison, even study methodology in this paper. In the paper, there are several places, that the authors use others results, which turns out the analysis and models for the paper is somewhat trivial, especially for node and edge distribution analysis.*

In [1], the scheduling and the routing are fixed and the emphasis was on understanding the scaling laws of the throughput. In Dousse's work on the percolation of SINR graphs, they consider nodes which are full-duplex (i.e., transmit and receive at the same time) and use percolative arguments to study the connectivity of the static SIR graph. Neither [1] nor Dousse are concerned with the path formation delay. In practice the radios are not full-duplex and hence the instantaneous network graph is always disconnected. In a wireless network the set of transmitting nodes changes almost at every time instant because of the MAC scheduler. In this paper, we consider an ad hoc network with half-duplex radios which uses multihop routing and slotted ALOHA for the network MAC contention. We introduce a random dynamic multi-digraph that models the connectivity of a wireless network more appropriately than a static graph. The main motivation of the paper is to introduce the concept of dynamic graphs and study their properties for the ALOHA MAC protocol. The connectivity graph formed is dynamic because of the scheduling and we are quantifying the minimum time required for a causal physical path to form between a pair of nodes. In the revised version we provide lower and upper bounds on the path formation time. We show that the time constant μ scales like $\beta\sqrt{\lambda}$. We use tools from epidemic processes and first passage percolation to study the delay.

2) *The proof of the major theorem is not clear. For example, the most significant definition is in equation (17). But the main conclusion does not follow smoothly even though we have Lemma 3 and Lemma 5 on page 13 and page 15, respectively. The paper could be better written and explained in this regard because this is the most important part in the paper.*

One of the major result is $\lim_{x\to\infty} T(o,x)/x = \mu$, i.e., linear scaling of the path formation time with respect to distance between the nodes. Also this result holds for any pair of nodes in the network. Another important contribution of the paper is the idea that the connectivity graph in an ad hoc wireless network is dynamic in nature and analyzing such a network using a static connectivity graph would lead to overly pessimistic results. For example the power that is required for a static SINR graph to percolate would be much higher than the power required for a path to form between any pair of nodes in a time that scales with the distance between the nodes. In practice this delay would be a lower bound to the actual delay for which one has to include the queueing delay.

In the revised version we have also included an upper bound and lower bound (25) (in this document) on the time constant (Lemma 4) rather than just showing that $0 < \mu < \infty$. We also have added more explanation and changed the notation so that the concepts are clearer and easier to understand in the revised version.

3) *Why define $\phi_t(k)$ and $\phi_r(k)$ in the system model? I did not find they are used later in the paper, except in the definition. The later notation $N_t(k)$ and $N_r(k)$ seem to be as the cardinality of the above two sets. But I do not understand why the authors define them again. If they are not the same, then what is the difference?*

$\phi_t(k)$ and $\phi_r(k)$ are used later in the paper, for example in Proposition 1, equations (6), (11). $N_t(x)$ denotes the number of receivers that a transmitter at x can connect to and $N_r(x)$ denotes the number of transmitters that a receiver at x can connect to. These are not equal to $|\phi_t(k)|$ and $|\phi_r(k)|$ respectively. Actually $|\phi_t(k)| = \infty$ and $|\phi_r(k)| = \infty$ almost surely for ALOHA parameter $p \in (0, 1)$.

4) *It is confusing sometime, we consider the distance from the origin to a location x and sometime we consider from location x and y.*

The point process ϕ is stationary and when we consider only a single transmitter-receiver pair, we can assume that the transmitter is at the origin and the receiver at x. When we are considering a path, we sometimes use x for the transmitter and y for the receiver. The usage depends on the context and the technical feasibility.

5) *What is β in $B(u, \beta\|x - y\|)$? I cannot find the definition, though it is used heavily in the paper for proofs and explanations.*

$\beta > 0$ is first introduced in Section 2: System model. We have clarified the definition of β in the revised version. In the scenario without fading and when the path-loss model is given by $\|x\|^{-\alpha}$, $\beta = (T^{1/\alpha} - 1)$ when the SINR threshold is given by $T > 1$.

6) *If graph $G(m, n)$ is formed between time instant m and n, is it relevant to the sequence? If not , then the definition of "causal path" is not clear. What does not mean " if the weights form a strictly increasing sequence?" How to determine weights? Under what conditions that the weights can form a sequence? in that regards? Timing?*

Consider the (single)-snapshot graph $g(k) = (\phi, E_k)$ with vertex set ϕ and edge set E_k formed at time instant k. All the edges of this graph have edge weights k (the time instant in which they were formed) and are directed. Now consider the multigraph $G(m, n) = (\phi, \cup_{k=n}^{m} E_k)$ and consider any directed path P from $x \in \phi$ to $y \in \phi$ (if it exists). Each edge in this path has a weight (the time it was formed) and we say that the path is casual if the edge weights are strictly increasing (in the direction of the path).

By a sequence we did not mean an infinite sequence. We have clarified the definition in the revised version in Section 2.

7) *The paper seems to implicitly refer to wireless networks. Then what is the relationship with transmitter and receivers? Each node is assumed to be full duplex or not? Why there can exist multiple nodes in one location? What kind of location /position is used in this paper? The paper did say that Dousse's work ignores the interference problem and said that it will be considered in the in this paper. However, throughout the reading, I cannot find interference being taken into account, except in the simulation results (Figure 3) which I don't understand neither.*

The paper deals explicitly with wireless networks.

1) The nodes (radios) are distributed spatially as a Poisson point process on the plane with density λ. This is introduced in Section 2.
2) All the nodes are half-duplex. A node chooses to transmit with probability p (ALOHA) in any time slot. If a node is not transmitting we assume it is in receiving mode. This is introduced in Section 2.
3) We use the modified protocol model to decide if a transmitter at x can connect to a receiver at y. The protocol model is a simple model that accounts for interference and a detailed discussion about this model can be found in [11]. For example if there is no fading and SINR$(x, y) > T$ with path loss model $\|x\|^{-\alpha}$ implies that $B(x, (T^{1/\alpha} - 1)\|x - y\|)$ does not contain any other transmitter. This is included in Section 2.
4) Our results about the path formation time are robust with the model, i.e., the result $T(o, x)/x \to \mu$ still holds true in the SIR model (physical model). We have used protocol model since it leads to clearer analytical

results. In [20], we have analyzed the dynamic connectivity graph with the SIR model and Rayleigh fading.

Reviewer 3 Comments:

1) *The main issue I have with the paper is with the model. The model they consider is the so-called "protocol model," used in reference [1] (Gupta and Kumar). However, they take only one of the two conditions from [1], namely that: "a transmitting node located at* x *can connect to a receiver located at y if the disk* $B(y, \beta\|y - x\|)$ *does not contain any other transmitting node." But they do not consider the other condition, which is a power constraint limit (in [1],* $\|y - x\| < R$ *(*)).*

This has several consequences. The first one is that it denatures the ALOHA protocol by making possible connections that actually are impossible. Consider the following set-up: on the line, put $x1$ *at* 0, $y1$ *at* 1, $x2$ *at* 4, $y2$ *at* 8, *and* $\beta = 1.5$, *where* xi *stands for a transmitter and* yi *for a receiver. The disk* $B(y1, 1.5)$ *is the line segment* $[-0.5, 2.5]$ *and contains no other point. The disk* $B(y2, 6)$ *is the line segment* $[2, 14]$ *and contains no other point. So according to the paper model,* $x1$ *and* $x2$ *are allowed to transmit at the same time. However, receiver* $y1$ *is closer to* $x2$ *than its actual intended receiver* $y2$, *and thus* $y1$ *will receive a signal from* $x2$ *strong enough to be decoded, and strong enough to interfere with that of x1.*

This issue arises in the graph in the paper. Take for instance Figure 2: The node at (approximate) coordinates $(1, -5)$ *can simultaneously transmit alongside the node at about* $(0, -4)$, *even though the transmission of the first node would drown the signal of the second. Because there is no power constraint, an interferer can still be outside of the disk* $B(y, \beta\|y - x\|)$ *yet still impact the communication between* y *and* x.

However, imposing a condition such as (*) would significantly alter the results of the paper: For instance, the average degree of the nodes is, in the current draft, independent on the density of the nodes. Imposing (*) would make the intensity of the PPP intervene and would render the results in this paper moot.

1) Regarding the example provided: The example provided by the reviewer is not a valid counterexample, and y_1 is fine as y_1 is able to receive from x_1. The connectivity between a transmitter and a receiver using the protocol model depends on the ratio of the distances between the transmitter-receiver and the receiver-nearest-interferer (SIR). y_1 will not be able to decode x_2 since $B(y_1, 1.5\|y_1 - x_2\|) = [-3.5, 5.5]$ contains another transmitter x_1. As you have indicated y_1 will be able to decode x_1 since it is closer to it than the nearest interferer x_2. Also in a modified Protocol model we are inherently assuming (if we want to connect it with the SIR model) that all the nodes transmit with equal power. So an interferer outside of the disk $B(y, \beta\|y - x\|)$ cannot dramatically affect the communication between y and x.
2) In the revised version we indicate that we are not using the exact protocol model. Using a constraint like $\|y - x\| < R$ is essentially a power constraint. In a practical system a transmitter chooses the power so that it is able to combat the path loss to the receiver. So in essence the power would be reduced with the density of the nodes. Since the distances between the nodes scale like $1/\sqrt{\lambda}$, if we set $R = k/\sqrt{\lambda}$ for some $k > 0$, the average in-degree does not depend on λ.
3) We also certainly agree that neglecting noise would lead to very optimistic results when $p \to 0$. When p is small the system is in the noise-limited regime, otherwise the system is in the interference-limited regime. When p is small, interference becomes very small (and since we are neglecting noise) and a transmitter can theoretically connect to a receiver very far from it. We have chosen not to use the power constraint so that the analysis is cleaner - well aware that any conclusions drawn for $p \to 0$ will be modelling artifacts and bear no relevance for practical systems.
4) If we consider the power constraint, i.e., $\|y - x\| < R$, then there is no guarantee that the time constant $\mu < \infty$ ($\mu > 0$ will still hold). We conjecture the following:
   If R is chosen so that the disc graph formed by placing discs of radius R around each node of ϕ percolates, i.e., $R > R_c(\lambda)$ then $\mu < \infty$.

2) *The paper would gain in clarity by stating explicitly some assumptions. For instance, it defines the contention mechanism as a slotted ALOHA with parameter* p. *But I should not be wondering if p stands for the probability of transmitting, or of receiving. This should be spelled out (it is, but on page 9, not page 3). Also confusing is the use of the letter* ϕ *(which looks like the empty set) as a set name.*

p represents the probability of transmitting. We have clarified the assumptions explicitly in the revised version in Section 2. ϕ is a standard symbol for representing point processes (the empty set is denoted by $\emptyset$), and that is the reason we have used it.

3)*Definition 1 is not clear at all: the weights it mentions are never formally defined. (the sentence which mention the weights is confusing: "Also each edge has a time stamp when it is formed as its weight.") Definition 1 states the graph is causal is the weights form a strictly increasing sequence. But that is not true! It should be the existence of an increasing subsequence that defines causality, since the edges are dynamic in the graph. Anyhow, the weights are never properly explained and are not even included in (2).*

The weight on each edge is the time at which the edge is formed. Definition 1 states that a *path* is casual if the weights form a strictly increasing sequence. By this we mean that the weights on the edges of the path should be increasing. We have clarified the definition in the revised version in Section 2.

4)

- *Replace "even" with "event" p4.*
- *P11: "So we are not counting loops." This sentence should be part of the previous one, maybe?*
- *The paper could refer some work in Delay Tolerant Networks. In particular, there are some similar results linking the delay and the distance in the area of DTNs.*

These are corrected in the revised version. We agree that our model is slightly related to the ones used in DTNs. However, that line of work focuses (to the best of our knowledge) on intermittent connectivity due to mobility of node failures, for example,
1) *"P. Jacquet, B. Mans, G. Rodolakis, F. Rocquencourt, and A. Australia, Information Propagation Speed in Delay Tolerant Networks: Analytic Upper Bounds, Information Theory, 2008. ISIT 2008. IEEE International Symposium on." 2008."* and
2) *" F. De Pellegrini, D. Miorandi, I. Carreras, and I. Chlamtac, A Graph-Based Model for Disconnected Ad Hoc Networks, in INFOCOM 2007. 26th IEEE International Conference on Computer Communications. IEEE, pp. 373-381, 2007."*
Further, we have not been able to find any work that considers interference, so we do not see a need to discuss the DTN literature.

5) *The simulations are nice, but do they refer to an actual simulation of the interference environment, or are they just a simulation of the mathematical model presented here? I could not find a description of the air interface, so I assume it's the latter, but it would be worth to have a real ALOHA system being used with some more realistic physical layer description to see how far off the model is.*

The simulations are based on the modified protocol model used in the paper. In [20], we have analyzed the dynamic connectivity graph with the SIR model and show some simulation results. We find similar linear scaling of the time required for a connection to form between two nodes with respect to the distance.