Interference and Outage in Clustered Wireless Ad Hoc Networks

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Abstract

In the analysis of large random wireless networks, the underlying node distribution is almost ubiquitously assumed to be the homogeneous Poisson point process. In this paper, the node locations are assumed to form a *Poisson clustered process* on the plane. We derive the distributional properties of the interference and provide upper and lower bounds for its CCDF. We consider the probability of successful transmission in an interference limited channel when fading is modeled as Rayleigh. We provide a numerically integrable expression for the outage probability and closed-form upper and lower bounds. We show that when the transmitter-receiver distance is large, the success probability is greater than that of a Poisson arrangement. These results characterize the performance of the system under geographical or MAC-induced clustering. We obtain the maximum intensity of transmitting nodes for a given outage constraint, *i.e.*, the transmission capacity (of this spatial arrangement) and show that it is equal to that of a Poisson arrangement of nodes. For the analysis, techniques from stochastic geometry are used, in particular the probability generating functional of Poisson cluster processes, the Palm characterization of Poisson cluster processes and the Campbell-Mecke theorem.

I. Introduction

A common and analytically convenient assumption for the node distribution in large wireless networks is the homogeneous (or stationary) Poisson point process (PPP) of intensity λ, where the number of nodes in a certain area of size A is Poisson with parameter λA, and the numbers of nodes in two disjoint areas are independent random variables. For sensor networks, this assumption is usually justified by claiming that sensor nodes may be dropped from aircraft in large numbers; for mobile ad hoc networks, it may be argued that terminals move independently from each other. While this may be the case for certain networks, it is much more likely that the node distribution is not "completely spatially random" (CSR), *i.e.*, that nodes

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are either clustered or more regularly distributed. Moreover, even if the complete set of nodes constitutes a PPP, the subset of *active* nodes (*e.g.*, transmitters in a given time-slot or sentries in a sensor network), may not be homogeneously Poisson. Certainly, it is preferable that simultaneous transmitters in an ad hoc network or sentries in a sensor network form more regular processes to maximize spatial reuse or coverage respectively. On the other hand, many protocols have been suggested that are based on clustered processes. This motivates the need to extend the rich set of results available for PPPs to other node distributions. The clustering of nodes may be due to geographical factors, for example communicating nodes inside a building or groups of nodes moving in a coordinated fashion. The clustering may also be "artificially" induced by MAC protocols. We denote the former as geographical clustering and the latter as logical clustering.

A. *Related Work*

There exists a significant body of literature for networks with Poisson distributed nodes. In [1] the characteristic function of the interference was obtained when there is no fading and the nodes are Poisson distributed. They also provide the probability distribution function of the interference as an infinite series. Mathar et al., in [2], analyze the interference when the interference contribution by a transmitter located at x, to a receiver located at the origin is exponentially distributed with parameter $\|x\|^2$. Using this model they derive the density function of the interference when the nodes are arranged as a one dimensional lattice. Also the Laplace transform of the interference is obtained when the nodes are Poisson distributed.

It is known that the interference in a planar network of nodes can be modeled as a shot noise process. Let $\{x_j\}$ be a point process in $\mathbb{R}$. Let $\{\beta_j(.)\}$ be a sequence of independent and identically distributed random functions on $\mathbb{R}^d$, independent of $\{x_j\}$. Then a generalized shot noise process can be defined as [3]

$$Y(x) = \sum_j \beta_j(x - x_j)$$

If $\beta_j()$ is the path loss model with fading, $Y(x)$ is the interference at location x if all nodes x_j are transmitting. The shot noise process is a very well studied process for noise modeling. It was first introduced by Schottky in the study of fluctuations in the anode current of a thermionic diode and it was studied in detail by Rice [4], [5]. Daley in 1971 defined multi-dimensional shot noise and examined its existence when the points $\{x_j\}$ are Poisson distributed in $\mathbb{R}^d$. The existence of generalized shot-noise process, for any point process was studied by Westcott in [3]. Westcott also provides the Laplace transform of the shot-noise when the points $\{x_j\}$ are distributed as a Poisson cluster process. Normal convergence of the multidimensional shot-noise process is shown by Heinrich and Schmidt [6]. They also show that when the points $\{x_j\}$ form a Poisson point process of intensity λ, the rate of convergence to a normal distribution is $\sqrt{\lambda}$.

In [7], Ilow and Hatzinakos model the interference as a shot noise process and show that the interference is a symmetric α-stable process [8] when the nodes are Poisson distributed on the plane. They also show that channel randomness affects the dispersion of the distribution, while the path-loss exponent affects the exponent of the process. The throughput and outage in the presence of interference are analyzed in [9]–[11].

In [9], the shot-noise process is analyzed using stochastic geometry when the nodes are distributed as Poisson and the fading is Rayleigh. In [12] upper and lower bounds are obtained under general fading and Poisson arrangement of nodes.

Even in the case of the PPP, the interference distribution is not known for all fading distributions and all channel attenuation models. Only the characteristic function or the Laplace transform of the interference can be obtained in most of the cases. The Laplace transform can be used to evaluate the outage probabilities under Rayleigh fading characteristics [9], [13]. In the analysis of outage probability, the *conditional* Laplace transform is required, *i.e.*, the Laplace transform given that there is a point of the process located at the origin. For the PPP, the conditional Laplace transform is equal to the unconditional Laplace transform. To the best of our knowledge, we are not aware of any literature pertaining to the interference characterization in a clustered network.

[14] introduces the notion of *transmission capacity*, which is a measure of the area spectral efficiency of the successful transmissions resulting from the optimal contention density as a function of the link distance. Transmission capacity is defined as the product of the maximum density of successful transmissions and their data rate, given an outage constraint. Weber et al., provide bounds for the transmission capacity under different models of fading, when the node location are Poisson distributed.

B. Main contributions and organization of the paper

In this work, we model the transmitters as a Poisson cluster process. To circumvent technical difficulties we assume that the receivers are not a part of this clustered process. We then focus on a specific transmit-receive pair at a distance R apart, see Fig 1. We evaluate the Laplace transform of the interference on the plane conditioned on the event that there is a transmitter located at the origin. Upper and lower bounds are obtained for the CCDF of the interference. From these bounds, it is observed that the interference is a heavy-tailed distribution with exponent $2/\alpha$ when the path loss function is $\|x\|^{-\alpha}$. When the path-loss function has no singularity at the origin (*i.e.*, remains bounded), the distribution of interference depends heavily on the fading distribution. Using the Laplace transform, the probability of successful transmission between a transmitter and receiver in an interference-limited Rayleigh channel is obtained. We provide a numerically integrable expression for the outage probability and closed-form upper and lower bounds. The *clustering gain* $G(R)$ is defined as the ratio of success probabilities of the clustered process and the PPP with the same intensity. It is observed that when the transmitter-receiver distance R is large, the clustering gain $G(R)$ is greater than unity and becomes infinity as $R \to \infty$. The gain $G(R)$ at small R depends on the path loss model and the total intensity of transmissions. We provide conditions on the total intensity of transmitters under which the gain is greater than unity for small R. This is useful to determine when logical clustering performs better than uniform deployment of nodes. We also obtain the maximum intensity of transmitting nodes for a given outage constraint, *i.e.*, the transmission capacity [12], [14], [15] of this spatial arrangement and show that it is equal to that of a Poisson arrangement of nodes. We observe that in

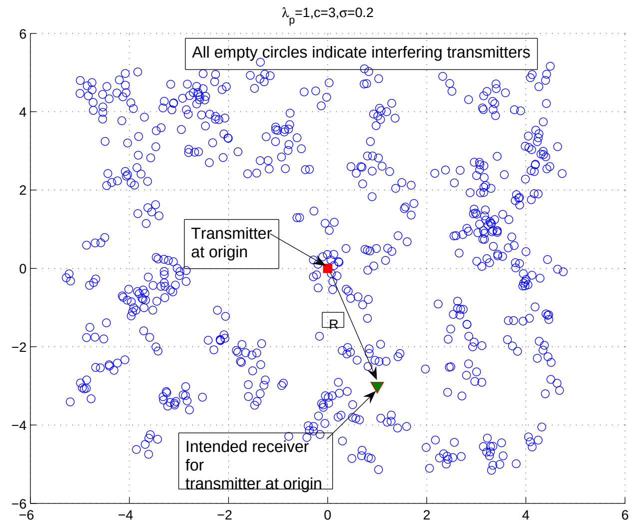

Fig. 1. Illustration of transmitters and receivers. Cluster density is 1. Transmitter density in each cluster is 3. Spread of each cluster is Gaussian with standard deviation $\sigma = 0.25$. Observe that the intended receiver for the transmitter at the origin is not a part of the cluster process. The transmitter at the origin is a part of the cluster located around the origin.

a spread-spectrum system, clustering is beneficial for long range transmissions, and we compare DS-CDMA and FH-CDMA.

The paper is organized as follows: in Section II we present the system model and assumptions, introduce the Neyman-Scott cluster process and derive its conditional generating functional. In Section III we derive the properties of interference, outage probability and the gain function $G(R)$. In Section IV, we derive the transmission capacity of the clustered network.

II. System Model and Assumptions

In this section we introduce the system model and derive some required results for the Poisson cluster process.

A. *System model and notation*

The location of transmitting nodes is modeled as a stationary and isotropic Poisson cluster process ϕ on $\mathbb{R}^2$. The receiver is not considered a part of the process. See Figure 1. Each transmitter is assumed to transmit at unit power. The power received by a receiver located at z due to a transmitter at x is modeled as $h_x g(x-z)$, where h_x is the power fading coefficient (square of the amplitude fading coefficient) associated with the channel between the nodes x and z. We also assume that all the fading coefficients are independent and are drawn from the same distribution. We will sometimes use h to denote a random variable that is i.i.d with the power fading coefficients. Let $\{o\}$ denote the origin $(0,0)$. We assume that the path loss model $g(x):\ \mathbb{R}^2 \setminus \{o\} \rightarrow \mathbb{R}^+$ satisfies the following conditions.

1) $g(x)$ is a continuous, positive, non-increasing function of $\|x\|$ and

$$\int_{\mathbb{R}^2\setminus B(o,\epsilon)} g(x)\mathrm{d}x < \infty, \quad \forall \epsilon > 0$$

where $B(o,\epsilon)$ denotes a ball of radius ϵ around the origin.

2)

$$\lim_{\|x\|\to\infty} \frac{g(x)}{g(x-y)} = 1, \quad \forall y \in \mathbb{R}^2 \tag{1}$$

$g(x)$ is usually taken to be a power law in the form $\|x\|^{-\alpha}$, $(1+\|x\|^{\alpha})^{-1}$ or $\min\{1, \|x\|^{-\alpha}\}$. To satisfy condition 1, we require $\alpha > 2$. The interference at node z on the plane is given by

$$I_\phi(z) \quad = \quad \sum_{x\in\phi} h_x g(x-z) \tag{2}$$

The conditions required for the existence of $I_\phi(z)$ are discussed in [3]. Let W denote the additive Gaussian noise the receiver. We say that the communication from a transmitter at the origin to a receiver situated at z is successful if and only if

$$\frac{hg(z)}{W + I_{\phi\setminus\{x\}}(z)} \quad \geq \quad T \tag{3}$$

or equivalently,

$$\frac{hg(z)}{W + I_\phi(z)} \geq \frac{T}{1+T}$$

For the calculation of outage probability and transmission capacity, the amplitude fading $\sqrt{h_x}$ is assumed to be Rayleigh with mean μ, but some results are presented for the more general case of Nakagami-m fading. Hence the powers h_x are exponentially and gamma distributed respectively. We will be evaluating the performance of spread-spectrum in some sections of the paper. Even though we evaluate spread-spectrum systems (specifically DS-CDMA and FH-CDMA) we will not be using any power control, the reason being that there is no central base station.

Notation: If $\lim_{x\to\infty} f(x)/g(x) = C$, we shall use $f(x) \sim g(x)$ if $C = 1$, $f(x) \lesssim g(x)$ if $0 < C < 1$ and $f(x) \gtrsim g(x)$ if $1 < C < \infty$.

B. Neyman-Scott cluster processes

Neyman-Scott cluster processes [16] are Poisson cluster processes that result from homogeneous independent clustering applied to a stationary Poisson process, where the parent points form a stationary Poisson process $\phi_p = \{x_1, x_2, \dots\}$ of intensity λ_p. The clusters are of the form $N^{x_i} = N_i + x_i$ for each $x_i \in \phi_p$. The N_i are a family of identical and independently distributed finite point sets with distribution independent of the parent process. The complete process ϕ is given by

$$\phi \quad = \quad \bigcup_{x\in\phi_p} N^x. \tag{4}$$

Note that the parent points themselves are not included. The daughter points of the representative cluster N_0 are scattered independently and with identical distribution $F(A) = \int_A f(x)\mathrm{d}x,\ A \subset \mathbb{R}^2$, around the

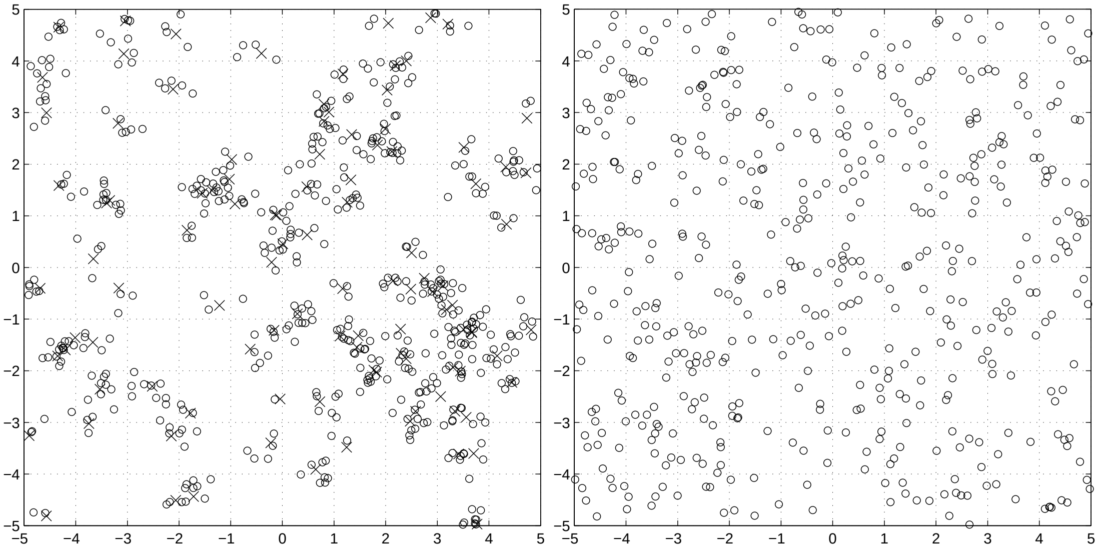

Fig. 2. (Left) Thomas cluster process with parameters $\lambda_p = 1, \bar{c} = 5$ and $\sigma = 0.2$. The crosses indicate the parent points. (Right) PPP with the same intensity $\lambda = 5$ for comparison.

origin. We also assume that the scattering density of the daughter process $f(x)$ is isotropic. This makes the process ϕ isotropic. The intensity of the cluster process is $\lambda = \lambda_p \bar{c}$, where $\bar{c}$ is the average number of points in representative cluster.

We further focus on more specific models for the representative cluster, namely Matern cluster processes and Thomas cluster processes. In these processes the number of points in the representative cluster is Poisson distributed with mean $\bar{c}$. For the Matern cluster process each point is uniformly distributed in a ball of radius a around the origin. So the density function $f(x)$ is given by

$$f(x) \quad = \quad \begin{cases} \frac{1}{\pi a^2}, & \|x\| \le a \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In the Thomas cluster process each point is scattered using a symmetric normal distribution with variance σ^2 around the origin. So the density function $f(x)$ is given by

$$f(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right).$$

A Thomas cluster process is illustrated in Fig.1. Newman-Scott cluster processes are also a Cox processes [16] when the number of points in the daughter cluster are Poisson distributed. The density of the driving random measure in this case is

$$\pi(y) = \bar{c} \sum_{x \in \phi_p} f(y - x)$$

Let $E_0^!(.)$ denote the expectation with respect to the reduced Palm measure [16], [17]. It is basically the conditional expectation for point processes, given the there is a point of the process at the origin but without including the point. Let $v(x):\mathbb{R}^2\to[0,1]$ and $\int_{\mathbb{R}^2}|1-v(x)|\mathrm{d}x<\infty$. When ϕ is Poisson of intensity λ, the conditional generating functional is

$$\begin{aligned} E_0^!\left(\prod_{x\in\phi} v(x)\right) &= E\left(\prod_{x\in\phi} v(x)\right) \\ &= \exp\left(-\lambda\int_{\mathbb{R}^2}[1-v(x)]\mathrm{d}x\right) \end{aligned} \tag{6}$$

The generating functional $\tilde{G}(v)=E\left(\prod_{x\in\phi} v(x)\right)$ of the Neyman-Scott cluster process is given by [16], [18]

$$\tilde{G}(v)=\exp\left(-\lambda_p\int_{\mathbb{R}^2}\left[1-M\left(\int_{\mathbb{R}^2} v(x+y)f(y)\mathrm{d}y\right)\right]\mathrm{d}x\right)$$

where $M(z)=\sum_{i=0}^{\infty}p_n z^n$ is the moment generating function of the number of points in the representative cluster. When the number of points in the representative cluster is Poisson with mean $\bar{c}$, as in the case of Matern and Thomas cluster processes,

$$M(z)=\exp(-\bar{c}(1-z)).$$

The generating functional for the representative cluster $G_c(v)$ is given by [18], [19]

$$G_c(v)=M\left(\int_{\mathbb{R}^2} v(x)f(x)\mathrm{d}x\right)$$

The reduced Palm distribution $P_0^!$ of a Neyman-Scott cluster process ϕ is given by [16]–[18], [20]

$$P_0^!=P*\tilde{\Omega}_0^! \tag{7}$$

where P is the distribution of ϕ, and $\tilde{\Omega}_0^!$ is the reduced Palm distribution of the finite representative cluster process N_0. "$*$" denotes the convolution of distributions, which corresponds to the superposition of ϕ and N_0. The reduced Palm distribution $\tilde{\Omega}_0^!$ is given by

$$\tilde{\Omega}_0^!(Y)=\frac{1}{\bar{c}}E\Big(\sum_{x\in N_0} 1_Y(\phi_{-x}\setminus\{0\})\Big) \tag{8}$$

where $\phi_x=\phi+x$, is a translated point process. We require the following lemma to evaluate the conditional Laplace transform of the interference. Let $\mathcal{G}(v)$ denote the conditional generating functional of the Neyman-Scott cluster process, *i.e.*,

$$\mathcal{G}(v)=E_0^!\left(\prod_{x\in\phi} v(x)\right) \tag{9}$$

We will use a dot to indicate the variable which the functional is acting on. For example $\mathcal{G}(v(\cdot-y))=E_0^![\prod_{x\in\phi} v(x-y)]$.

Lemma 1: Let $0\le v(x)\le 1$. The conditional generating functional of Thomas and Matern clustered processes is

$$\mathcal{G}(v)=\tilde{G}(v)\int_{\mathbb{R}^2} G_c(v(\cdot-y))f(y)\mathrm{d}y.$$

Proof: Let $Y_x = Y + x$. From (8), we have

$$\tilde{\Omega}^!_0(Y) \quad = \quad \frac{1}{\bar{c}} E\Big(\sum_{x \in N_0} 1_{Y_x}(\phi \setminus \{x\})\Big) \tag{10}$$

Let $\Omega()$ denote the probability distribution of the representative cluster. Using the Campbell-Mecke theorem [16], we get

$$\begin{aligned} \tilde{\Omega}^!_0(Y) &= \frac{1}{\bar{c}} \int_{\mathbb{R}^2} \int_{\mathcal{N}} 1_{Y_x}(\phi) \Omega^!_x(\mathrm{d}\phi) \bar{c} F(\mathrm{d}x) \\ &= \int_{\mathbb{R}^2} \int_{\mathcal{N}} 1_{Y_x}(\phi) \Omega^!_x(\mathrm{d}\phi) f(x) \mathrm{d}x \end{aligned} \tag{11}$$

Here $\mathcal{N}$ denotes the space of locally finite and simple point sequences [16] on $\mathbb{R}^2$. Since the representative cluster has a Poisson distribution of points, by Slivnyak's theorem [16] we have $\Omega^!_x(.) = \Omega(.)$. Hence

$$\begin{aligned} \tilde{\Omega}^!_0(Y) &= \int_{\mathbb{R}^2} \int_{N} 1_{Y_x}(\phi) \Omega(\mathrm{d}\phi) f(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \Omega(Y_x) f(x) \mathrm{d}x \end{aligned} \tag{12}$$

For notational convenience let ψ denote N_0. Let $\psi_y = \psi + y$. Using (7), we have

$$\begin{aligned} \mathcal{G}(v) &= \int_{\mathcal{N}} \int_{\mathcal{N}} \prod_{x \in \phi \cup \psi} v(x) P(\mathrm{d}\phi) \tilde{\Omega}^!_0(\mathrm{d}\psi) \\ &= \int_{\mathcal{N}} \prod_{x \in \phi} v(x) P(\mathrm{d}\phi) \int_{\mathcal{N}} \prod_{x \in \psi} v(x) \tilde{\Omega}^!_0(\mathrm{d}\psi) \\ &= \tilde{G}(v) \int_{\mathcal{N}} \prod_{x \in \psi} v(x) \tilde{\Omega}^!_0(\mathrm{d}\psi) \qquad (13) \\ &\stackrel{(a)}{=} \tilde{G}(v) \int_{\mathcal{N}} \prod_{x \in \psi} v(x) \int_{\mathbb{R}^2} \Omega(\mathrm{d}\psi_y) f(y) \mathrm{d}y \\ &= \tilde{G}(v) \int_{\mathbb{R}^2} \int_{\mathcal{N}} \prod_{x \in \psi} v(x) \Omega(\mathrm{d}\psi_y) f(y) \mathrm{d}y \\ &= \tilde{G}(v) \int_{\mathbb{R}^2} \int_{\mathcal{N}} \prod_{x \in \psi} v(x - y) \Omega(\mathrm{d}\psi) f(y) \mathrm{d}y \\ &\stackrel{(b)}{=} \tilde{G}(v) \int_{\mathbb{R}^2} G_c(v(\cdot - y)) f(y) \mathrm{d}y \end{aligned}$$

(a) follows from (12), and (b) follows from the definition of $\mathcal{G}(.)$. ■

So from the above lemma, we have

$$\begin{aligned} \mathcal{G}(v) &= \exp\left(-\lambda_p \int_{\mathbb{R}^2} \Big[1 - M\Big(\int_{\mathbb{R}^2} v(x+y) f(y) \mathrm{d}y\Big)\Big] \mathrm{d}x\right) \\ &\times \int_{\mathbb{R}^2} M\Big(\int_{\mathbb{R}^2} v(x-y) f(x) \mathrm{d}x\Big) f(y) \mathrm{d}y \end{aligned} \tag{14}$$

The above equation holds when all the integrals are finite. Since $f(x) = f(-x)$, then $\int_{\mathbb{R}^2} v(x+y) f(y) \mathrm{d}y = \int_{\mathbb{R}^2} v(x-y) f(y) \mathrm{d}y = v * f$, so

$$\mathcal{G}(v) \quad = \quad \exp\left(-\lambda_p \int_{\mathbb{R}^2} \Big[1 - M((v * f)(x))\Big] \mathrm{d}x\right) \int_{\mathbb{R}^2} M((v * f)(y)) f(y) \mathrm{d}y \tag{15}$$

Likelihood and nearest neighbor functions of the Poisson cluster process, which involve similar calculations with Palm distributions are provided in [21]. One can obtain the nearest-neighbor distribution function of Thomas or Matern cluster process as $D(r) = \mathcal{G}(1_{B(o,r)^c}(.))$. In some cases the number of points per cluster may be fixed rather than Poisson. The conditional generating functional, for this case is given in Appendix B.

III. Interference and Outage Probability of Poisson Cluster Processes

In this section, we first derive the characteristics of interference in a Poisson clustered process conditioned on the existence of a transmitting node at the origin. We then evaluate the outage probability for a transmit-receive pair when the transmitters are distributed as a Neyman-Scott cluster process, with the number of points in each cluster is Poisson with mean $\bar{c}$ and density function $f(x)$.

A. *Properties of the Interference* $I_\phi(z)$

Let $\mathcal{L}_h(s)$ denote the Laplace transform of the fading random variable h.

Lemma 2: The conditional Laplace transform of the interference is given by

$$\mathcal{L}_{I_\phi(z)}(s) = \mathcal{G}\left(\mathcal{L}_h(sg(\cdot - z))\right) \tag{16}$$

Proof: From (2) we have

$$\begin{aligned} \mathcal{L}_{I_\phi(z)}(s) &= E_0^! \exp(-s\sum_{x\in\phi} h_x g(x-z)) \\ &= E_0^! \left[\prod_{x\in\phi} \exp(-sh_x g(x-z))\right] \\ &\stackrel{(a)}{=} E_0^! \left[\prod_{x\in\phi} \mathcal{L}_h(sg(x-z))\right] \end{aligned} \tag{17}$$

where (a) follows from the independence of h_x and (16) follows from (9). ■

We observe from Lemma 2 and (15), that the conditional Laplace transform of the interference $\mathcal{L}_{I_\phi(z)}(s)$ depends on the position z. This implies that the distribution of the interference depends on the location z at which we observe the interference. This is in contrast to the fact that the interference distribution is independent of the location z when the transmitters are Poisson distributed on the plane [9], [12]. This is due to the non-stationarity of the reduced Palm measure of the Neyman-Scott cluster processes. If one interprets $I_\phi(z)$ as a stochastic process, it is then a non stationary process due to the above reason.

Let $\mathcal{K}_n(B)$ denote the reduced n-th factorial moment measure [16], [18] of a point process ψ, and let $B = B_1 \times \ldots \times B_{n-1},\ B_i \in \mathbb{R}^2$.

$$\mathcal{K}_n(B) = E_0^! \left[\sum_{x_1,\ldots,x_{n-1}\in\psi}^{x_i\neq x_j} 1_B(x_1,\ldots,x_{n-1})\right] \tag{18}$$

$\mathcal{K}_2(B(0,R))$, for example, denotes the average number of points inside a ball of radius R centered around the origin, given that a point exists at the origin. First and second moments of the interference can be determined using the second and third order reduced factorial moments. The average interference (conditioned on the event that there is a point of the process at the origin) is given by

$$\begin{aligned} E_0^![I_\phi(z)] &= E_0^!\left[\sum_{x\in\phi} h_x g(x-z)\right] \\ &= E[h]\lambda\int_{\mathbb{R}^2} g(x-z)\mathcal{K}_2(\mathrm{d}x) \end{aligned} \tag{19}$$

Since the process ϕ is stationary, $\mathcal{K}_2(B)$ can be expressed as [16], [22]

$$\mathcal{K}_2(B) = \frac{1}{\lambda^2}\int_B \rho^{(2)}(x)\mathrm{d}x,$$

where $\rho^{(2)}(x)$ is the second order product density[1]. So we have

$$E_0^![I_\phi(z)] = \frac{E[h]}{\lambda}\int_{\mathbb{R}^2} g(x-z)\rho^{(2)}(x)dx \tag{20}$$

Example: Thomas Cluster Process. In this case, from [16]

$$\frac{\rho^{(2)}(x)}{\lambda^2} = 1 + \frac{1}{4\pi\lambda_p\sigma^2}\exp\left(\frac{-\|x\|^2}{4\sigma^2}\right)$$

where $\lambda = \lambda_p\bar{c}$. We obtain

$$E_0^![I_\phi(z)] = EI_{\text{Poi }(\lambda)} + \frac{\bar{c}E[h]}{4\pi\sigma^2}\int_{\mathbb{R}^2} g(x-z)\exp\left(\frac{-\|x\|^2}{4\sigma^2}\right)\mathrm{d}x \tag{21}$$

Where $EI_{\text{Poi }(\lambda)}$ is the average interference seen by a receiver located at z, when the nodes are distributed as a PPP with intensity λ. The above expression also shows that the mean interference[2] is indeed larger than for the PPP. One can also get the above from the conditional Laplace transform in Lemma 2 and using $E_0^![I_\phi(z)] = -\frac{d}{ds}\mathcal{L}_{I_\phi(z)}(s)|_{s=0}$. In the following theorem we provide bounds to the tail probability of the interference $I_\phi(z)$ for *any stationary distribution* ϕ of transmitters. We adapt the technique presented in [15] to derive the tail bounds of the interference. We denote the tail probability (CCDF) of $I_\phi(z)$ by $\bar{F}_I(y) = \mathbb{P}(I_\phi(z) \geq y)$.

Theorem 1: When the transmitters are distributed as a stationary and isotropic point process ϕ of intensity λ with conditional generating functional $\mathcal{G}$ and second order product density $\rho^{(2)}$, the tail probability $\bar{F}_I(y)$ of the interference at location z, conditioned on a transmitter present at the origin[3] is lower bounded by

[1]Intuitively, this indicates the probability that there are two points separated by $\|x\|$. For PPP, it is $\rho^{(2)}(x) = \lambda^2$ independent of x. Also the second order product density is a function of two arguments *i.e.*, $\rho^{(2)}(x_{1,}, x_2)$. But when the process ϕ is stationary, $\rho^{(2)}$ depends only on the difference of its arguments *i.e.*, $\rho^{(2)}(x_1, x_2) = \nu(x_1 - x_2)$ for all $x_1, x_2 \in \mathbb{R}^2$. Furthermore if ϕ is motion-invariant, i.e., stationary and isotropic, then ν depends only on $\|x_1 - x_2\|$ [16, pg 112].

[2]Note that for $g(x) = \|x\|^{-\alpha}$, $E_0^![I_\phi(z)]$ is diverging.

[3]We do not include the contribution of the transmitter at the origin in the interference. This is because the transmitter at the origin is the intended transmitter which we focus on.

$\bar{F}_I^l(y)$ and upper bounded by $\bar{F}_I^u(y)$, where

$$\bar{F}_I^l(y) = 1 - \mathcal{G}\left(F_h\left(\frac{y}{g(.-z)}\right)\right) \tag{22}$$

$$\bar{F}_I^u(y) = 1 - (1-\varphi(y))\mathcal{G}\left(F_h\left(\frac{y}{g(.-z)}\right)\right) \tag{23}$$

where $F_h(x)$ denotes the CDF of the power fading coefficient h and

$$\varphi(y) = \frac{1}{y\lambda}\int_{\mathbb{R}^2} g(x-z)\rho^{(2)}(x)\int_0^{y/g(x-z)} \nu \mathrm{d}F_h(\nu)\mathrm{d}x.$$

Proof: The basic idea is to divide the transmitter set into two subsets ϕ_y and ϕ_y^c where,

$$\phi_y = \{x \in \phi,\ h_x g(x-z) > y\} \tag{24}$$

$$\phi_y^c = \{x \in \phi,\ h_x g(x-z) \leq y\} \tag{25}$$

ϕ_y consists of those transmitters, whose contribution to the interference exceeds y. We have $I_\phi(z) = I_{\phi_y}(z) + I_{\phi_y^c}(z)$, where $I_{\phi_y}(z)$ corresponds to the interference due to the transmitter set ϕ_y and $I_{\phi_y^c}(z)$ corresponds to the interference due to the transmitter set ϕ_y^c. Hence we have

$$\begin{aligned}\bar{F}_I(y) &= \mathbb{P}(I_{\phi_y}(z) + I_{\phi_y^c}(z) \geq y) \\ &\geq \mathbb{P}(I_{\phi_y}(z) \geq y) \\ &= 1 - \mathbb{P}(I_{\phi_y}(z) < y) \\ &= 1 - \mathbb{P}(\phi_y = \emptyset).\end{aligned} \tag{26}$$

We can evaluate the probability $\mathbb{P}(\phi_y = \emptyset)$ that ϕ_y is empty using the conditional Laplace functional as follows:

$$\begin{aligned}\mathbb{P}(\phi_y = \emptyset) &= E_0^! \prod_{x\in\phi} 1_{h_x g(x-z)\leq y} \\ &\stackrel{(a)}{=} E_0^! \prod_{x\in\phi} E_{h_x}\left(1_{h_x g(x-z)\leq y}\right) \\ &= E_0^! \prod_{x\in\phi} F_h\left(\frac{y}{g(x-z)}\right) \\ &= \mathcal{G}\left(F_h\left(\frac{y}{g(\cdot-z)}\right)\right),\end{aligned} \tag{27}$$

where (a) follows from the independence of h_x. To obtain the upper bound

$$\begin{aligned}\bar{F}_I(y) &= \mathbb{P}(I_\phi > y \mid I_{\phi_y} > y)\bar{F}_I^l(y) + \mathbb{P}(I_\phi > y \mid I_{\phi_y} \leq y)(1-\bar{F}_I^l(y)) \\ &\stackrel{(a)}{=} 1 - \mathcal{G}\left(F_h\left(\frac{y}{g(\cdot-z)}\right)\right) + \mathbb{P}(I_\phi > y \mid I_{\phi_y} \leq y)\mathcal{G}\left(F_h\left(\frac{y}{g(\cdot-z)}\right)\right) \\ &= 1 - (1-\mathbb{P}(I_\phi > y \mid I_{\phi_y} \leq y))\mathcal{G}\left(F_h\left(\frac{y}{g(\cdot-z)}\right)\right)\end{aligned} \tag{28}$$

where (a) follows from the lower bound we have established. To evaluate $\mathbb{P}(I_\phi > y \mid I_{\phi_y} \leq y)$ we use the Markov inequality (the Chebeshev inequality can also be used but is more difficult to be evaluated in this particular setting). We have

$$\begin{aligned}
\mathbb{P}(I_\phi > y \mid I_{\phi_y} \leq y) &= \mathbb{P}(I_\phi > y \mid \phi_y = \emptyset) \\
&\overset{(a)}{\leq} \frac{E_0^!\left(I_\phi \mid \phi_y = \emptyset\right)}{y} \\
&= \frac{1}{y} E_0^! \sum_{x \in \phi} h_x g(x-z) 1_{h_x g(x-z) \leq y} \\
&= \frac{1}{y} E_0^! \sum_{x \in \phi} g(x-z) \int_0^{y/g(x-z)} \nu \mathrm{d}F_h(\nu) \\
&\overset{(b)}{=} \frac{1}{y\lambda} \int_{\mathbb{R}^2} g(x-z) \int_0^{y/g(x-z)} \nu \mathrm{d}F_h(\nu) \rho^{(2)}(x) \mathrm{d}x \qquad (29)
\end{aligned}$$

(a) follows from the Markov inequality, and (b) follows from a procedure similar to the calculation of the mean interference in (20). ■

In the proof of Lemma 3, we show $\varphi(y) \sim \theta_2 y^{-2/\alpha}$ when $g(x) = \|x\|^{-\alpha}$. This indicates the tightness of the bounds for large y. Lemma 3 shows that the interference is a heavy-tailed distribution with parameter $2/\alpha$ when the nodes are distributed as a Neyman-Scott cluster process.

Lemma 3: For $g(x) = \|x\|^{-\alpha}$, the lower and upper bounds to CCDF $\bar{F}_I(y)$ of the interference at location z, when the nodes are distributed as a Neyman-Scott cluster process scale as follows for $y \to \infty$.

$$\bar{F}_I^l(y) \sim \theta_1 y^{-2/\alpha} \qquad (30)$$

$$\bar{F}_I^u(y) \sim (\theta_1 + \theta_2) y^{-2/\alpha} \qquad (31)$$

where $\theta_1 = \pi \bar{c}[(f * f)(z) + \lambda_p] \int_0^\infty \nu^{2/\alpha} \mathrm{d}F_h(\nu)$ and $\theta_2 = 2\theta_1/(\alpha - 2)$.

Proof: See Appendix A. ■

Remarks:

1) Observe that $\theta_1 = \pi\lambda^{-1}\rho^{(2)}(z) \int_0^\infty \nu^{2/\alpha} \mathrm{d}F_h(\nu)$. A similar kind of scaling law with $\theta_1 = \pi\lambda^{-1}\rho^{(2)}(z) E_h[\nu^{2/\alpha}]$ and $\theta_2 = 2\theta_1/(\alpha-2)$ can be obtained when the transmitters are scattered as any "nice"[4] stationary, isotropic point process with intensity λ and second order product density $\rho^{(2)}(x) \neq 0$ at $x = z$.
2) A similar heavy-tailed distribution with parameter $2/\alpha$ was obtained for Poisson interference in [1], [15]. Since $2/\alpha < 1$, the mean and hence the variance diverge. This can also be inferred from (21) and is due to the singularity of the channel function $g(x) = \|x\|^{-\alpha}$ at the origin. For Matern cluster processes

[4] We require the conditional generating functional to have a series expansion with respect to reduced n-th factorial moment measures of the reduced Palm distribution [22] similar to that of the expansion of generating functional [16, p.116] and [23]. The proof of the existence and the series expansion of the conditional generating functional with respect to reduced n-th factorial moment measures, would be of more technical nature following a technique used in [23]. If such an expansion exists it is straightforward to prove the scaling laws for the CCDF of interference similar to Lemma 3, with $\theta_1 = \pi\lambda^{-1}\rho^{(2)}(z)E_h[\nu^{2/\alpha}]$ and $\theta_2 = 2\theta_1/(\alpha-2)$.

$(f * f)(z) = 0$, for $\|z\| > 2a$ and for Thomas cluster processes $(f * f)(z)$ is a Gaussian with variance $2\sigma^2$. Hence for large z, we observe that the constants θ_1 become similar to that of the unconditional interference. This is because, the contribution of the cluster at origin becomes small as we move far from the origin.

3) When the path loss function is $g(x) = (1 + \|x\|^\alpha)^{-1}$, the distribution of the interference more strongly depends on the fading model. Using a similar proof as in Lemma 3, one can deduce an exponential tail decay when $g(x) = (1 + \|x\|^\alpha)^{-1}$ and Rayleigh fading. Similarly if the power fading coefficient

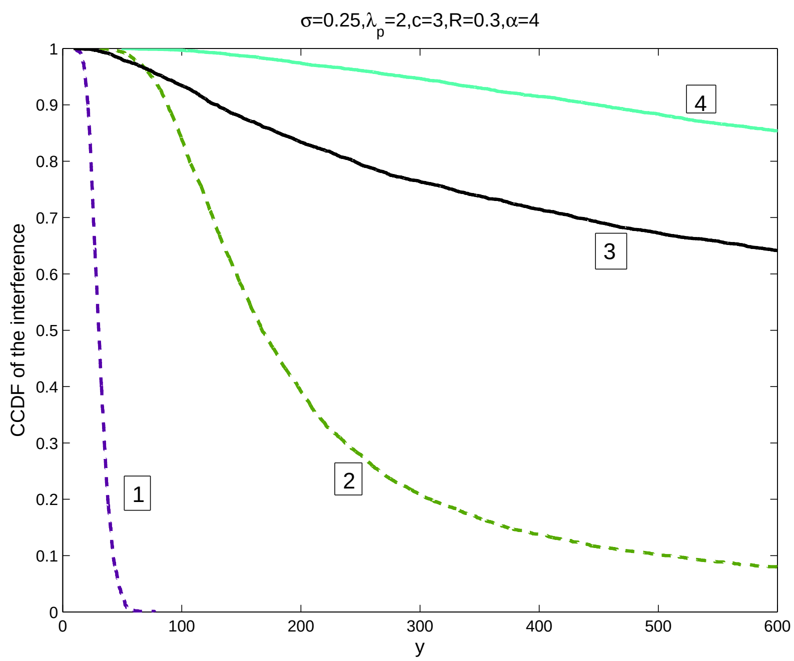

Fig. 3. $\lambda_p = 2, \bar{c} = 3, \sigma = 0.25, \alpha = 4, R = 0.3$: Comparison of the interference CCDF for different path-loss models and different fading. They were generated using Monte-Carlo simulation. Curves #1 and #2 correspond to $g(x) = (1 + \|x\|^\alpha)^{-1}$. Curve #1 corresponds to Rayleigh fading and exhibits an exponential decay. Curve #2 for which h is distributed as generalized Pareto with parameters $k = 1, \theta = 0, \sigma_p = 1$ (a hypothetical power fading distribution which exhibits power law decay) exhibits a power law decay. Curves #3 (generalized Pareto) and #4 (Rayleigh) correspond to $g(x) = \|x\|^{-\alpha}$ and exhibit a heavy tail for both fading distributions.

follows a power-law distribution with exponent k, the tail of the interference shows a power-law decay. This is because of the presence of the term $y^{-2/\alpha} \int_y^\infty [1 - F_h(u)](u - y)^{2/\alpha - 1} \mathrm{d}u$ in the proof. So when using non-singular channel models, the interference has a more intricate dependence on the fading characteristics rather than a simple dependence on $E_h[\nu^{2/\alpha}]$ as in the singular case. This behavior is well understood for Poisson and unconditional Poisson cluster shot noise process [24], [25]. The properties of interference for different path loss models with no fading, when the nodes are uniformly distributed are discussed in [26].

B. Success probability: $\mathbb{P}(success)$

Let the desired transmitter be located at the origin and the receiver at location z at distance $R = \|z\|$ from the transmitter. With a slight abuse of notation we shall be using R to denote the point $(R, 0)$. The probability of success for this pair is given by

$$\mathbb{P}(\text{success}) = \mathbb{P}^{!0}\Big(\frac{hg(z)}{W + I_\phi(z)} \geq T\Big) \tag{32}$$

We now assume Rayleigh fading, *i.e.*, the received power is exponentially distributed with mean μ. So we have

$$\begin{aligned}\mathbb{P}(\text{success}) &= \int_0^\infty e^{-\mu s T/g(z)} \mathrm{d}\mathbb{P}(W + I_{\phi\setminus\{0\}}(z) \leq s) \\ &= \mathcal{L}_{I_\phi(z)}(\mu T/g(z))\mathcal{L}_W(\mu T/g(z))\,, \end{aligned} \tag{33}$$

When h_x is Rayleigh we have

$$\mathcal{L}_h(sg(x-z)) \quad = \quad \frac{\mu}{\mu + sg(x-z)} \tag{34}$$

At $s = \mu T/g(R)$ we observe that the above expression will be independent of the mean of the exponential distribution μ.

Lemma 4: [Success probability] The probability of successful transmission between the transmitter at the origin and the receiver located at $z \in \mathbb{R}^2$, when $W \equiv 0$ (no noise), is given by

$$\begin{aligned}\mathbb{P}(\text{success}) = &\underbrace{\exp\Big\{-\lambda_p \int_{\mathbb{R}^2} \Big[1 - \exp(-\bar{c}\beta(z,y))\Big]\mathrm{d}y\Big\}}_{T_1} \\ &\times \underbrace{\int_{\mathbb{R}^2} \exp(-\bar{c}\beta(z,y)) f(y)\mathrm{d}y}_{T_2}\end{aligned} \tag{35}$$

where

$$\beta(z,y) = \int_{\mathbb{R}^2} \frac{g(x-y-z)}{\frac{g(z)}{T} + g(x-y-z)} f(x)\mathrm{d}x \tag{36}$$

Proof: Follows from (34) and Lemma 2. ■

The success probability, when the number of nodes in each cluster is fixed is given in the Appendix B. See Figure 4 for comparison. When the fading is Nakagami-m, the probability of success is evaluated in the Appendix C for integer m.

Remarks:

1) The term T_1 in (35) captures the interference without the cluster at the origin (*i.e.*, without conditioning); it is independent[5] of the position z since the original cluster process is stationary (can be verified by change of variables $y_1 = y + z$). The second term T_2 is the contribution of the transmitter's

[5]By this we mean the unconditional interference distribution which leads to this term does not depend on the location z. The term T_1 does depend on $g(z)$.

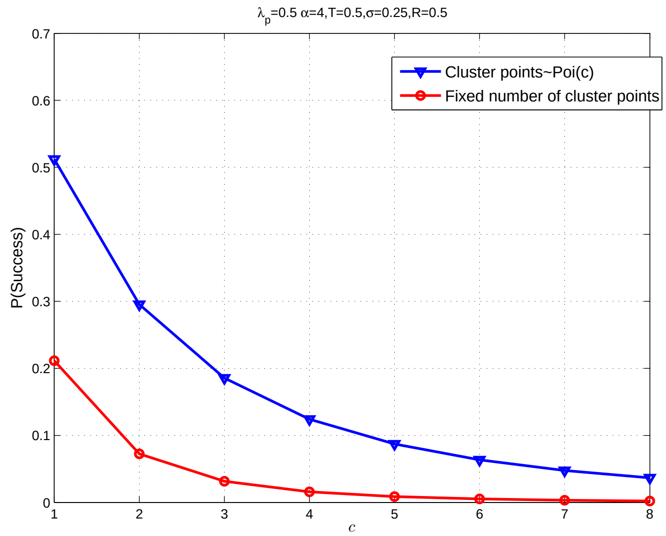

Fig. 4. Comparison of $\mathbb{P}$(success) when the number of points in a cluster are fixed and Poisson distributed with parameter $\bar{c}$.

cluster; it is identical for all z with $\|z\| = R$ since f and g are isotropic. So the success probability itself is the same for all z at distance R. This is because the Palm distribution is always isotropic when the original distribution is motion-invariant [16]. Hence we shall use $\beta(R, y)$ to denote $\beta(z, y)$ where $z = Re^{i\theta}$. We shall also use R and $(R, 0)$ interchangeably and will be clear by the context.

2) From the above argument we observe that $\mathbb{P}$(success) depends only on $\|z\| = R$ and not on the angle of z. So the success probability should be interpreted as an average over the circle $\|z\| = R$, *i.e.*, the receiver may be uniformly located anywhere on the circle of radius R around the origin. For large distances R, there is a very high probability that the receiver is located in an empty space and not in any cluster. Hence for large R the success probability is higher than that of a PPP of the same intensity. If the receiver is also conditioned to be in a cluster, we have to multiply (at least heuristically) by a term that is similar to T_2 and this would significantly reduce the success probability.

3) From Lemma 4, we have $\mathbb{P}(\text{success}) = E_0^![\exp(-sI_\phi(z))]$ evaluated at $s = \mu T/g(z)$. If $\mu T/g(z)$ is *small,* and $\int g(z)dz < \infty$ *(i.e,* finite average interference) then, $\mathbb{P}(\text{success}) \leq P_p(\lambda)$. This follows from (21) and the fact that $E_0^![I_\phi(z)]$ is the slope of the curve $E_0^![\exp(-sI_\phi(z))]$ at $s = 0$. This implies that at small distances, spread spectrum (DS-CDMA) works better with a Poisson distribution of nodes. (If the distance R is large, then the spreading gain has to increase approximately like $g(z)$ to keep $\mu T/g(z)$ small.)

4) Let M be the DS-CDMA spreading factor. We have $\mathbb{P}(\text{outage}) = \mathbb{P}(I_\phi(z) > \frac{M}{T} h_x g(z))$. For $g(x) = \|x\|^{-\alpha}$, we have the following scaling law for the outage probability with respect to the spreading gain.

$$\theta_1 R^2 M^{-2/\alpha} T^{2/\alpha} E_h[\nu^{-2/\alpha}] \overset{(a)}{\lesssim} \mathbb{P}(\text{outage}) \overset{(b)}{\lesssim} \frac{\alpha}{\alpha - 2} \theta_1 R^2 M^{-2/\alpha} T^{2/\alpha} E_h[\nu^{-2/\alpha}], \tag{37}$$

where $E_h[\nu^{-2/\alpha}] = \int_0^\infty \nu^{-2/\alpha} \mathrm{d}F_{h^2}(\nu)$. (a) and (b) follow from Lemma 3. Also observe that these scaling bounds are valid for any fading distribution for which $E_h[\nu^{-2/\alpha}] < \infty$. Similar scaling laws with the exponent of M being $-2/\alpha$ can be obtained when the transmitters are Poisson distributed on the plane. When the fading is Rayleigh *i.e.*, $h \sim \exp(\mu)$, the lower bound is

$$\pi\bar{c}[(f*f)(R)+\lambda_p]\Gamma\Big(1+\frac{2}{\alpha}\Big)\Gamma\Big(1-\frac{2}{\alpha}\Big)R^2M^{-2/\alpha}T^{2/\alpha}$$

and the upper bound is $\alpha/(\alpha-2)$ times the lower bound. $\Gamma(z)$ represents the standard Gamma function.

We now derive closed form upper and lower bounds on $\mathbb{P}(\text{success})$.

Lemma 5: [Lower bound]

$$\mathbb{P}(\text{success}) \geq P_p(\lambda)P_p(\bar{c}\hat{f}^*) \tag{38}$$

where $P_p(\lambda)$ denotes the success probability when ϕ is a PPP, $\hat{f}^* = \sup_{y\in\mathbb{R}^2}(f*f)(y)$, and $\lambda = \lambda_p\bar{c}$.

Proof: The first factor in (35), T_1 can be lower bounded by the success probability in the standard PPP $P_p(\lambda)$, and the second factor can be lower bounded by $P_p(\bar{c}\hat{f}^*)$. From (35) and the fact that $1-\exp(-\delta x) \leq \delta x, \delta \geq 0$, we have

$$\begin{aligned}\mathbb{P}(\text{success}) \quad &\geq \quad \underbrace{\exp\Big(-\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,y)\mathrm{d}y\Big)}_{\text{Term1}} \\ &\quad\quad \times \underbrace{\int_{\mathbb{R}^2}\exp(-\bar{c}\beta(R,y))f(y)\mathrm{d}y}_{\text{Term2}}\end{aligned} \tag{39}$$

$$\begin{aligned}\text{Term1} \quad &= \quad \exp\Big(-\lambda\int_{\mathbb{R}^2}\beta(R,y)\mathrm{d}y\Big) \\ &\overset{(a)}{=} \quad \exp\Big(-\lambda\int_{\mathbb{R}^2}\frac{g(y)}{\frac{g(R)}{T}+g(y)}\mathrm{d}y\Big) \\ &= \quad P_p(\lambda)\end{aligned} \tag{40}$$

(a) follows from change of variables, interchanging integrals and using $\int f(x) = 1$.

$$\text{Term2} \quad = \quad \int_{\mathbb{R}^2}\exp(-\bar{c}\beta(R,y))f(y)\mathrm{d}y$$

Since $\exp(-x)$ is convex and $f(x) > 0, \int f(x) = 1$, Using Jensen's inequality $(Ef(x) \geq f(E(x)))$ we have,

$$\text{Term2} \quad \geq \quad \exp\Big(-\bar{c}\int_{\mathbb{R}^2}\beta(R,y)f(y)\mathrm{d}y\Big)$$

Changing variables and using $f(x) = f(-x)$,we get,

$$\begin{aligned}\text{Term2} \quad &\geq \quad \exp\Big(-\bar{c}\int_{\mathbb{R}^2}\frac{g(x)}{\frac{g(R)}{T}+g(x)}\int_{\mathbb{R}^2}f(x+z-y)f(y)\mathrm{d}y\mathrm{d}x\Big) \\ &\geq \quad \exp\Big(-\bar{c}\int_{\mathbb{R}^2}\frac{g(x)}{\frac{g(R)}{T}+g(x)}(f*f)(x+z)\mathrm{d}x\Big)\end{aligned} \tag{41}$$

Hence

$$\text{Term2} \quad \geq \quad P_p(\bar{c}\hat{f}^*) \tag{42}$$

■

Since $f \in L_p$, by Young's inequality [27] we have $\hat{f}^* \leq \|f\|_p \|f\|_q$, where $1/p+1/q = 1$ (conjugate exponents). For $a \geq 1/\sqrt{\pi}$ (Matern) and $\sigma \geq 1/\sqrt{2\pi}$ (Thomas), we get $\mathbb{P}(\text{success}) \geq P_p(\lambda)P_p(\bar{c})$. In general, $\hat{f}^* \leq \|f\|_\infty \|f\|_1$, which is $1/\pi a^2$ for Matern and $1/2\pi\sigma^2$ for Thomas processes. In the latter case, when f is Gaussian, $f * f$ is also Gaussian with variance $2\sigma^2$, hence $\hat{f}^* \leq 1/4\pi\sigma^2$. From [9], we get (by change of variables):

$$P_p(\lambda) = \exp\left(-\lambda \int_{\mathbb{R}^2} \beta(R,y)\mathrm{d}y\right). \tag{43}$$

We have

- for $g(x) = \|x\|^{-\alpha}$, $P_p(\lambda) = \exp(-\lambda R^2 T^{2/\alpha} C(\alpha))$ [9], where $C(\alpha) = \big(2\pi\Gamma(2/\alpha)\Gamma(1-2/\alpha)\big)/\alpha = \frac{2\pi^2}{\alpha}\csc(2\pi/\alpha)$.
- for $g(x) = (1+\|x\|^\alpha)^{-1}$, $P_p(\lambda) = \exp(-\lambda T C(\alpha)(T+g(R))^{2/\alpha-1} g(R)^{-2/\alpha})$.

Let $\beta_I = \int_{\mathbb{R}^2} \beta(R,y)\mathrm{d}y$, $\hat{\beta} = \sup_{y\in\mathbb{R}^2} \beta(R,y)$ and $\hat{f} = \sup_{y\in\mathbb{R}^2} f(y)$. By Hölders inequality we have $\hat{\beta} \leq \min\{1, \hat{f}\beta_I(R)\}$. Also let $\kappa = \int_{\mathbb{R}^2} \beta(R,y)f(y)\mathrm{d}y$.

Lemma 6: [Upper bound]

$$\mathbb{P}(\text{success}) \quad \leq \quad P_p\left(\frac{\lambda}{1+\bar{c}\hat{\beta}}\right) \tag{44}$$

Proof: Neglecting the second term T_2 and using $\exp(-\delta x) \leq 1/(1+\delta x)$, we have

$$\begin{aligned}
\mathbb{P}(\text{success}) &\leq \exp\Big(-\lambda_p \int_{\mathbb{R}^2} \Big[1 - \frac{1}{1+\bar{c}\beta(R,y)}\Big]\mathrm{d}y\Big) \\
&= \exp\Big(-\lambda_p \int_{\mathbb{R}^2} \frac{\bar{c}\beta(R,y)}{1+\bar{c}\beta(R,y)}\mathrm{d}y\Big) \\
&\leq \exp\Big(-\frac{\lambda_p \bar{c}}{1+\bar{c}\hat{\beta}} \int_{\mathbb{R}^2} \beta(R,y)\mathrm{d}y\Big)
\end{aligned} \tag{45}$$

■

From the above two lemmata, we get

$$P_p(\lambda)P_p(\bar{c}\hat{f}^*) \leq \quad \mathbb{P}(\text{success}) \quad \leq P_p\left(\frac{\lambda}{1+\bar{c}\hat{\beta}}\right) \tag{46}$$

from which follows $\mathbb{P}(\text{success}) \to P_p(\lambda)$ as $\frac{\bar{c}}{\sigma}, \frac{\bar{c}}{a} \to 0$ as expected. In Lemma 6, we have neglected the contribution of the transmitter's cluster. We derive the following upper bound in the proof of Lemma 8,

$$\mathbb{P}(\text{success}) \quad \leq \quad P_p(\lambda)\exp\left(\lambda\beta_I \nu(\bar{c}\hat{\beta})\right)\left[1 - \left(1-\nu(\bar{c}\hat{\beta})\right)\bar{c}\kappa\right] \tag{47}$$

where $\nu(x) = (\exp(-x) - 1 + x)/x$. Substituting for $\nu(x)$, we have

$$\mathbb{P}(\text{success}) \leq P_p\left(\frac{\lambda(1-\exp(-\bar{c}\hat{\beta}))}{\bar{c}\hat{\beta}}\right)\left[1 - \left(1-\exp(-\bar{c}\hat{\beta})\right)\frac{\kappa}{\hat{\beta}}\right] \tag{48}$$

(48) is a tighter bound than the bound in Lemma 6, but not easily computable due to the presence of κ (for a given R, T and σ, κ and β^* are constants). In (48), the outage due to the interference by the transmitting cluster is also taken into account.

The proof of Lemmata 5 and 6 also indicates that it is only by conditioning on the event that there is a point at the origin that the success probability of Neyman-Scott cluster processes can be lower than the Poisson process of the same intensity. This implies that the cluster around the transmitter causes the maximum "damage". So as the receiver moves away from the transmitter, the Neyman-Scott cluster process has a better success probability than the PPP. So, it is not true in general that cluster processes have a lower success probability than PPPs of the same intensity. For example from Figure 5, we see that for

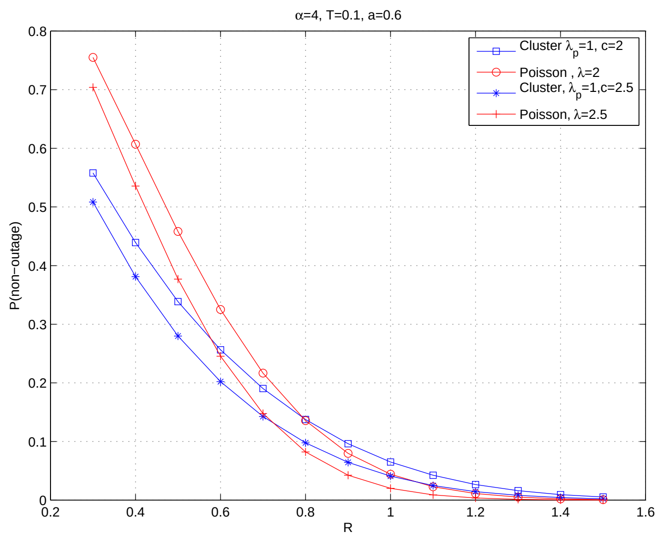

Fig. 5. Comparison of success probability for cluster and Poisson process of intensity 2

$R < 0.8$, the PPP has a better success probability than the Matern process. In Subsection III-C we give a more detailed analysis, which reveals that a PPP with intensity $\lambda_p \bar{c}$ has a lower success probability than a clustered process of the same intensity *for large transmit-receiver distances.* On the other hand, for small R, the success probability of the PPP is higher.

C. Clustering Gain $G(R)$

In this subsection we compare the performances of a clustered network and a Poisson network of the same intensity with Rayleigh fading. We deduce how the clustering gain depends on the transmitter receiver distance. We use the following notation,

$$P_1(R, \bar{c}, \lambda_p) \triangleq \exp\Big(- \lambda_p \int_{\mathbb{R}^2} 1 - \exp(-\bar{c}\beta(R, y))\mathrm{d}y \Big) \tag{49}$$

$$P_2(R, \bar{c}) \triangleq \int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R, y)) f(y)\mathrm{d}y \tag{50}$$

So $\mathbb{P}(\text{success}) = P_1(R, \bar{c}, \lambda_p) P_2(R, \bar{c})$. P_2 is the probability of success due to the presence of the cluster at the origin near the transmitter. P_1 is the probability of success in the presence of other clusters. Interference from these other clusters contributes more to the outage when R is large. This is also intuitive, since as the receiver moves away from the transmitting cluster, the interference from the other clusters starts to dominate. We define the *clustering gain* $G(R)$ as

$$G(R) = \frac{P_1(R, \bar{c}, \lambda_p) P_2(R, \bar{c})}{P_p(\lambda_p \bar{c})}$$

The fluctuation of $G(R)$ around unity indicates the existence of a crossover point R^* below which the PPP performs better than clustered process and vice versa. The values of $G(R)$ at the origin and infinity indicate the gain of scheduling transmitters as clusters instead of being spread uniformly on the plane. So it is beneficial to induce logical clustering of transmitters by MAC if $G(R) > 1$.

We first consider $G(R)$ for large R, *i.e.*, $\lim_{R\to\infty} G(R)$. By the dominated convergence theorem and (1), we have

$$\begin{aligned} \lim_{R\to\infty} P_2(R, \bar{c}) &= \int_{\mathbb{R}^2} \exp\Big(- \bar{c} \int_{\mathbb{R}^2} \lim_{R\to\infty} \frac{f(x)}{1 + \frac{g(R)}{Tg(x-y-R)}} \mathrm{d}x \Big) f(y) \mathrm{d}y \\ &= \exp\Big(\frac{-\bar{c}}{1 + 1/T} \Big) \end{aligned} \tag{51}$$

Also from the derivation of upper bound we have $P_1(R, \bar{c}, \lambda_p) \leq P_p\left(\frac{\lambda}{1+\bar{c}\hat{\beta}}\right)$. Hence from the definition of $P_p(x)$ we have, $\lim_{R\to\infty} P_1(R, \bar{c}, \lambda_p) = 0$. Hence for large R,

$$P_1(R, \bar{c}, \lambda_p) < P_2(R, \bar{c}) \tag{52}$$

So for large R, most of the damage is done by transmitting nodes other than the cluster in which the intended transmitter lies.

Lemma 7:

$$\lim_{R\to\infty} \frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} = 0 \tag{53}$$

Proof: See Appendix D ■

Hence for large R, $\frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} \leq \exp(\frac{-\bar{c}}{1+1/T})$. From (51) we have $P_p(\lambda_p \bar{c}) \leq P_1(R, \bar{c}, \lambda_p) P_2(R, \bar{c})$, for large R, *i.e.*, $G(R) > 1$ for large transmit-receive distance. We have $\lim_{R\to\infty} G(R) = \infty$. Hence the *Poisson point process with intensity $\lambda_p \bar{c}$, has a lower success probability than the clustered process of the same intensity for large transmit receiver distances.*

For small R, $G(R)$ depends on the behavior of the path loss function, $g(x)$ at $\|x\| = 0$. We consider the two cases when the channel function is singular at the origin or not.

1) $\lim_{\|x\|\to 0} g(x) = \infty$: In this case we observe that $G(0) = 1$. But at small R, $G(R)$ is less than 1. We have the following lemma.

Lemma 8: If $(f * f)(x) > \|x\|$ for *small* $\|x\|$ and $g(x) = \|x\|^{-\alpha}$, then for *small* R,

$$\mathbb{P}(\text{success}) \leq P_p(\lambda_p \bar{c}) \tag{54}$$

Proof: See Appendix E. ■

Note that $f(x)$ for Matern and Thomas cluster process have the required property. Hence when $g(x) = \|x\|^{-\alpha}$, *the PPP with intensity $\lambda_p \bar{c}$, has a higher success probability than the clustered process of the same intensity* *for small transmit receiver distance*. Lemma 8 and the fact that $G(\infty) = \infty$ also indicate the existence of a crossover point R^* between the success curves of the PPP and the cluster process. So it is not true in general that the performance of the clustered process is better or worse than that of the Poisson process. This is because, for the same intensity, a clustered process will have clusters of transmitters (where interference is high) and also vacant areas (where there are no transmitters and interference is low), whereas in a Poisson process, the transmitters are uniformly spread.

2) $\lim_{\|x\|\to 0} g(x) = \hat{g} < \infty$*:* $P_1(R,\bar{c},\lambda_p)$ can be written as

$$P_1(R,\bar{c},\lambda_p) = P_p(\lambda_p\bar{c})\exp\Big(\lambda_p \int_{\mathbb{R}^2} \underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}\bar{c}^n \beta(R,y)^n \, \mathrm{d}y}_{>0}\Big)$$

Hence $G(R)$ can also be written as follows

$$G(R) = P_2(R,\bar{c})\exp\left(\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,y)\eta(\bar{c},R,y)\mathrm{d}y\right) \tag{55}$$

where $\eta(\bar{c},R,y) = \nu(\bar{c}\beta(R,y))$, with $\nu(x) = (\exp(-x) - 1 + x)/x$. Observe that $0 \leq \eta(\bar{c},R,y) \leq 1, \forall x > 0$. If the total density of the transmitters is fixed *i.e.*, $\lambda = \lambda_p\bar{c}$ is constant, how does $G(R)$ behave with respect to $\bar{c}$? We have the following lemma which characterizes the monotonicity of $G(R)$ with respect to $\bar{c}$.

Lemma 9: Given $\lambda = \lambda_p\bar{c}$ is constant, $G(R)$ is decreasing with $\bar{c}$, *i.e.*, $\frac{\mathrm{d}G(R)}{\mathrm{d}\bar{c}} \leq 0$, $\forall \bar{c} > 0$ iff $\lambda \leq \lambda^*(R,T)$, where

$$\lambda^*(R,T) = \frac{2\int_{\mathbb{R}^2}\beta(R,y)f(y)\mathrm{d}y}{\int_{\mathbb{R}^2}\beta(R,y)^2\mathrm{d}y}$$

Proof: From (55),

$$\begin{aligned} G(R) &= P_2(R,\bar{c})\exp\left[\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,a)\eta(\bar{c},R,a)\mathrm{d}a\right] \\ &= \int_{\mathbb{R}^2}\exp\Big(-\bar{c}\beta(R,y) + \lambda\int_{\mathbb{R}^2}\beta(R,a)\eta(\bar{c},R,a)\mathrm{d}a\Big)f(y)\mathrm{d}y \end{aligned} \tag{56}$$

We have $\frac{\mathrm{d}\eta(\bar{c},R,z)}{\mathrm{d}\bar{c}}|_{\bar{c}=0} = \beta(R,z)/2$ and $\frac{\mathrm{d}\eta(\bar{c},R,z)}{\mathrm{d}\bar{c}}$ is decreasing in $\bar{c}$.

$$\begin{aligned} \frac{\mathrm{d}G(R)}{\mathrm{d}\bar{c}} &= \int_{\mathbb{R}^2}\left[-\beta(R,y) + \lambda\int_{\mathbb{R}^2}\beta(R,a)\frac{\mathrm{d}\eta(\bar{c},R,a)}{\mathrm{d}\bar{c}}\mathrm{d}a\right]\exp\Big(-\bar{c}\beta(R,y) + \lambda\int_{\mathbb{R}^2}\beta(R,a)\eta(\bar{c},R,a)\mathrm{d}a\Big)f(y)\mathrm{d}y \\ &= \exp[\lambda\int_{\mathbb{R}^2}\beta(R,a)\eta(\bar{c},R,a)\mathrm{d}a]\underbrace{\int_{\mathbb{R}^2}\Big[-\beta(R,y) + \lambda\int_{\mathbb{R}^2}\beta(R,z)\frac{\mathrm{d}\eta(\bar{c},R,a)}{\mathrm{d}\bar{c}}\mathrm{d}a\Big]\exp\Big(-\bar{c}\beta(R,y)\Big)f(y)\mathrm{d}y}_{T_2(\bar{c})} \end{aligned}$$

Since $\eta'(\bar{c},R,z)$ is decreasing in $\bar{c}$, we have $T_2(\bar{c})$ is decreasing in $\bar{c}$. So a necessary and sufficient condition for $\frac{dG(R)}{d\bar{c}} \leq 0 \; \forall \bar{c} > 0$ is $T_2(0) \leq 0$. We want

$$\begin{aligned} T_2(0) &= \int_{\mathbb{R}^2}\left[-\beta(R,y) + \frac{\lambda}{2}\int_{\mathbb{R}^2}\beta^2(R,z)\mathrm{d}z\right]f(y)\mathrm{d}y \leq 0 \\ &\Rightarrow \lambda \leq \frac{2\int_{\mathbb{R}^2}\beta(R,y)f(y)\mathrm{d}y}{\int_{\mathbb{R}^2}\beta^2(R,z)\mathrm{d}z} \end{aligned} \tag{57}$$

■

Remarks:

1) Since $\beta(0, y) \neq 0$, we have that, $G(0)$ is increasing with λ_p (like $\exp(\lambda_p)$), and hence will be greater than 1 at some λ_p for a fixed $\bar{c}$.
2) We have $\lim_{\bar{c} \to 0} G(R) = 1$ and specifically $G(0) = 1$ at $\bar{c} = 0$.
3) From Lemma 9 and Remark 2 we can deduce $G(R) < 1, \ \forall \bar{c} > 0$ if $\lambda < \lambda^*(R, T)$ *i.e.*, the gain $G(R)$ decreases from 1 with increasing $\bar{c}$ if the total intensity of transmitters is less than $\lambda^*(R, T)$.
4) Since $G(R)$ is continuous with respect to R, $G(R)$ is close to $G(0)$ for small R.
5) From Figure 7, we observe that $G(R)$ increases monotonically with R.

In Figure 6, $\lambda^*(0, T)$ is plotted against T. We provide some heuristics as to when logical clustering does not perform better than a uniform distribution of points:

- The exact value of R at which $G(R)$ crosses 1 is difficult to find analytically due to the highly nonlinear nature of $G(R)$. If such a crossover point exists (depends on the path-loss model) we will denote it by R^*.
- If $g(x) = \|x\|^{-\alpha}$, it is better to induce logical clustering by the MAC scheme if the link distance is larger than R^*. Otherwise it is better to schedule the transmissions so that they are scattered uniformly on the plane.
- If $g(0) < \infty$ and for a constant intensity $\lambda_p \bar{c}$, it is always beneficial to induce clustering for long-hop transmissions. When R is small the answer depends on the total intensity $\lambda_p \bar{c}$. If $\lambda_p \bar{c} < \lambda^*(0, T)$ then $G(0) < 1$ by observation 3, and hence $G(R) < 1$ for *small* R by observation 4. Also when $\lambda_p \bar{c} < \lambda^*(0, T)$, it is better to reduce logical clustering by decreasing $\bar{c}$ and increasing λ_p, since $G(0)$ is a decreasing function of $\bar{c}$. From Figure 6 we observe that $\lambda^*(0, 0.5) \approx 1.26$ when $g(x) = (1 + \|x\|^4)^{-1}$ and $\sigma = 0.25$. In Figure 7, $G(R)$ is plotted for $\lambda_p \bar{c} = 0.75, 9$ for the same values of σ, α and the same channel function as of Figure 6. When $\lambda_p \bar{c} = 9 > \lambda^*(0, 0.5)$, we observe that the gain curve $G(R)$ is approximately 10 at the origin and increases. When $\lambda_p \bar{c} = 0.75 < \lambda^*(0, 0.5)$, $G(R)$ starts around 0.25 and crosses 1 at $R \approx 1.2$. We also observe that $G(R)$, for the non-singular $g(x)$, seem to increase monotonically. We also observe that the gain function for $g(x) = \|x\|^{-\alpha}$ decreases from 1 initially and then increases to infinity.
- For DS-CDMA, the value of T is smaller by a factor equal to the spreading gain. From Figure 6, we observe that the threshold $\lambda^*(0, T)$ for clustering to be beneficial at small distances increases with decreasing T. Hence for a constant intensity of transmissions $\lambda_p \bar{c}$, the benefit of clustering decreases with increasing spreading gain for small link distances. So for DS-CDMA (for a large spreading gain) it is better to make the transmissions uniform on the plane for smaller link distances and cluster the transmitters for long-range communication.
- For FH-CDMA, the total number of transmissions $\lambda_p \bar{c}$ is reduced by the spreading gain while T remains constant (see Figure 6). Hence $\lambda_p \bar{c} < \lambda^*(0, T)$ for small distances and one can draw similar conclusions as that of DS-CDMA. The relative gain between FH-CDMA and DS-CDMA with clustering is more difficult to characterize analytically.

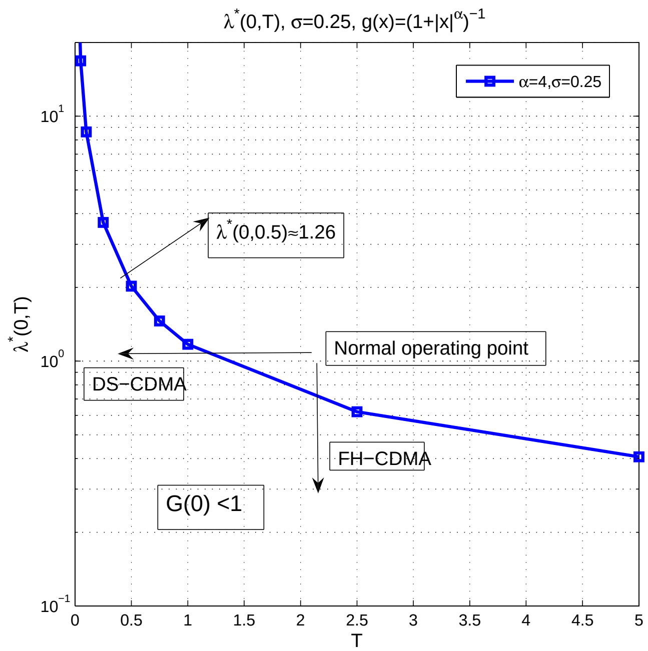

Fig. 6. $\lambda^*(0,T)$ versus T for $g(x)=(1+\|x\|^4)^{-1}$, $\sigma=0.25$. The region below the curve consists of all the pairs of $(T,\lambda=\lambda_p\bar{c})$ such that $G(0)<1$. "Normal operating point" denotes a pair (T,λ) that lies above the curve $(T,\lambda^*(0,T))$. Suppose we use FH-CDMA, the total intensity decreases by a factor of spreading gain and hence we move vertically downwards into the $G(0)<1$ region. If DS-CDMA is used, the threshold T decreases by a factor of spreading gain and hence we move horizontally towards the left into the $G(0)<1$ region.

IV. Transmission Capacity of Clustered Transmitters

It is important to understand the performance of ad hoc wireless networks. Transmission capacity was introduced in [12], [14], [15] and is defined as the product of the maximum density of successful transmissions and their data rate, given an outage constraint. More formally, if the intensity of the contending transmitters is λ with an outage threshold T and a bit rate b bits per second per hertz, then the transmission capacity at a fixed distance R is given by

$$C(\epsilon,T)=b(1-\epsilon)\sup_{\lambda}\{\lambda : P(\lambda,T)\geq 1-\epsilon\} \tag{58}$$

where $P(\lambda,T)$ denotes the success probability of a given transmitter receiver pair. More discussion about the transmission capacity and its relation to other metrics like transport capacity is provided in [15]. Note that the results proved in [12], [14], [15] are for Poisson arrangement of transmitters.

In this section we evaluate the transmission capacity when the transmitters are arranged as a Poisson clustered process. We prove that for small values of ϵ, the transmission capacity of the clustered process coincides with that of the Poisson arrangement of nodes. We also show that care should be taken in defining transmission capacity for general distribution of nodes. For notational convenience we shall assume $b=1$. For the clustered process, $P(\lambda,T)$ denotes the success probability of the cluster process with intensity $\lambda=\lambda_p\bar{c}$

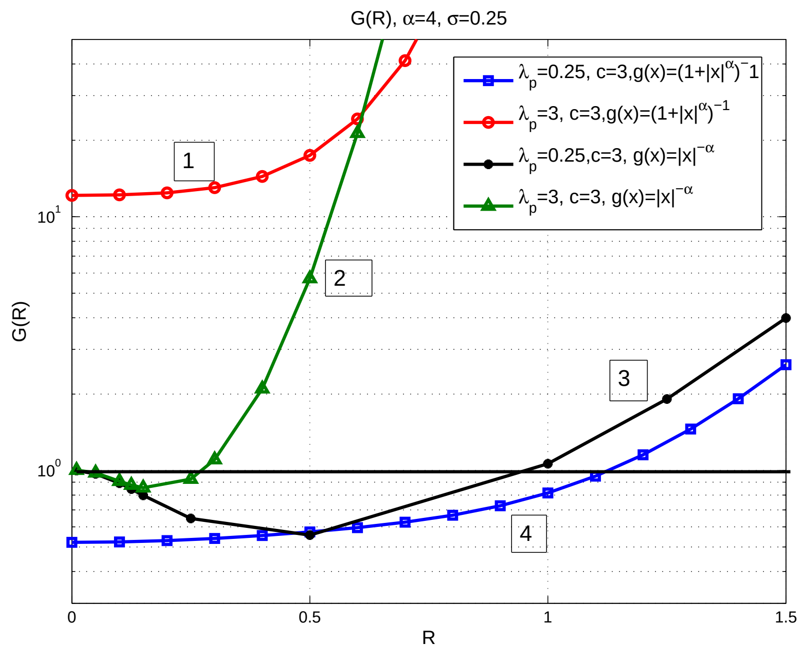

Fig. 7. $G(R)$ versus R, $\alpha = 4, \sigma = 0.25$. Observe that the gain curves #2 and #3, which correspond to the singular channel, start at 1 decrease and then increase above unity. For the gain curve #4, the total intensity of transmitters is $3 * 0.25 = 0.75$ which is less than the threshold $\lambda^*(0, 0.5) \approx 1.26$. Hence the gain curve for this case starts below unity at $R = 0$ and then increases. For the gain curve #1 the total intensity is $9 > 1.26$. By chance, in the present case the gain curve #1 starts around 10 and increases.

and threshold T. Let $P_l(\lambda, T)$, $P_u(\lambda, T)$ denote lower and upper bounds of the success probability $P(\lambda, T)$ and the corresponding sets A_l, A_u defined by $A_\chi := \{\lambda : P_\chi(\lambda, T) \geq 1 - \epsilon\}$ for $\chi \in \{l, u\}$. We then have $A_l \subset A \subset A_u$ which implies

$$\sup A_l \leq \sup A \leq \sup A_u. \tag{59}$$

Let $C_l(\epsilon, T) = \sup A_l$ and $C_u(\epsilon, T) = \sup A_u$ denote lower and upper bounds to the transmission capacity.

For a PPP we have from (43), $P_p(\lambda, T) = \exp(-\lambda\beta_I)$ (β_I does not depend on λ). Hence the transmission capacity of a PPP denoted by $C_p(\epsilon, T)$ is given by

$$\begin{aligned} C_p(\epsilon, T) &= \frac{1-\epsilon}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) \\ &\approx \frac{\epsilon(1-\epsilon)}{\beta_I}, \ \epsilon \ll 1 \end{aligned} \tag{60}$$

For Neyman-Scott cluster processes, the intensity $\lambda = \lambda_p \bar{c}$. We first to try to consider both λ_p and $\bar{c}$ as optimization parameters for the transmission capacity, i.e.

$$C(\epsilon, T) \quad := \quad (1-\epsilon) \sup\{\lambda_p \bar{c} : \lambda_p > 0, \bar{c} > 0, \text{outage-constraint}\} \tag{61}$$

without individually constraining the parent node density or the average number of nodes per cluster.

Lemma 10: The transmission capacity of Poisson clustered processes is lower bounded by the transmission capacity of the PPP,

$$C(\epsilon, T) \geq C_l(\epsilon, T) = C_p(\epsilon, T) \tag{62}$$

Proof: From Lemma 5, we have $P_l(\lambda, T) = P_p(\lambda_p \bar{c}) P_p(\bar{c}\hat{f}^*)$. So to get a lower bound, from (59) we have to find

$$\sup\left\{\lambda_p \bar{c} : \lambda_p \bar{c} + \bar{c}\hat{f}^* \quad \leq \quad \frac{1}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) = \frac{C_p(\epsilon, T)}{1-\epsilon}\right\} \tag{63}$$

This maximum value of $\lambda_p \bar{c}$ is attained when, $\lambda_p \to \infty$, while $\bar{c} \to 0$, such that $\bar{c}\lambda_p = C_p(\epsilon, T)(1-\epsilon)^{-1}$. So we have $C_l(\epsilon, T) = C_p(\epsilon, T)$. ■

Also observe that $\lambda_p \to \infty$ and $\bar{c} \to 0$. This corresponds to the scenario in which the clustered process degenerated to a PPP. We also have the following upper bound.

Lemma 11: Let $\rho(T) = k/\hat{\beta}$ with $k = \int \beta(R, y) f(y) \mathrm{d}y$. For $\epsilon < 1 - e^{-\rho(T)}$, we have

$$C(\epsilon, T) \quad \leq \quad C_u(\epsilon, T) = C_p(\epsilon, T) \tag{64}$$

Proof: See Appendix F. ■

Theorem 2: For $\epsilon \leq 1 - e^{-\rho(T)}$ we have $C(\epsilon, T) = C_p(\epsilon, T)$.

Proof: Follows from the Lemmata 10 and 11. ■

From the above two proofs, when ϵ is small, the transmission capacity is equal to the Poisson process of same intensity. This capacity is achieved when $\lambda_p \to \infty$ and $\bar{c} \to 0$. This is the scenario in which the cluster process becomes a PPP. This is due to the definition of the transmission capacity as $C(\epsilon, T) := \sup\{\lambda_p \bar{c} : \lambda_p > 0, \bar{c} > 0, \text{outage-constraint}\}$ where we have two variables to optimize over.

Instead we may fix λ_p as constant and find the transmission capacity with respect to $\bar{c}$. So we define constrained transmission capacity as

$$C^*(\epsilon, T) := \lambda_p (1-\epsilon) \sup\{\bar{c} : \bar{c} > 0, \text{outage-constraint}\} \tag{65}$$

We have the following bounds for $C^*(\epsilon, T)$

Theorem 3:

$$\frac{\lambda_p C_p(\epsilon, T)}{\lambda_p + \hat{f}^*} \quad \leq C^*(\epsilon, T) \leq \quad \frac{\lambda_p C_p(\epsilon, T)}{\max\left\{0, \lambda_p - \frac{\hat{\beta}}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right)\right\}} \tag{66}$$

Proof: From the lower bound on $\mathbb{P}(\text{success})$, we have to find

$$\sup\left\{\bar{c} : \lambda_p \bar{c} + \bar{c}\hat{f}^* \quad \leq \quad \frac{1}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) = \frac{C_p(\epsilon, T)}{1-\epsilon}\right\} \tag{67}$$

So we have $C_l^*(\epsilon, T) = C_p(\epsilon, T)/(\hat{f}^* + \lambda_p)$.

From the upper bound on $P(\text{success})$,we have to find

$$\sup\left\{\bar{c} : \frac{\lambda_p \bar{c}}{1 + \bar{c}\hat{\beta}} \quad \leq \quad \frac{1}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) = \frac{C_p(\epsilon, T)}{1-\epsilon}\right\} \tag{68}$$

■

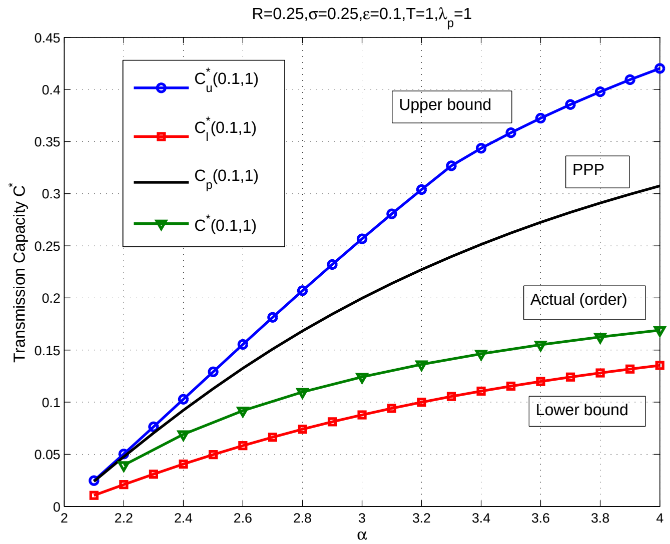

Fig. 8. Upper and lower bounds of $C^*(\epsilon, T)$ versus α, $g(x) = \|x\|^{-\alpha}, T = 1, \sigma = 0.25, \epsilon = 0.1, \lambda_p = 1$

One can also derive an order approximation to the constrained transmission capacity when ϵ is very small. We have the following order approximation to transmission capacity.

Proposition 1: When λ_p is fixed, the constrained transmission capacity is given by

$$C^*(\epsilon, T) \quad = \quad (1-\epsilon)\left(\frac{\epsilon\lambda_p}{\lambda_p\beta_I + k} + o(\epsilon)\right) \tag{69}$$

when $\epsilon \longrightarrow 0$.

Proof: Let $\gamma(\bar{c})$ denote the outage probability, *i.e.*,

$$\gamma(\bar{c}) \quad = \quad 1 - \exp\Big\{ - \lambda_p \int_{\mathbb{R}^2} 1 - \exp[-\bar{c}\beta(R,y)]\mathrm{d}y\Big\} \int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R,y))f(y)\mathrm{d}y \tag{70}$$

We have $\mathrm{d}\gamma(\bar{c})/\mathrm{d}\bar{c} > 0$, which implies $\gamma(\bar{c})$ is increasing and invertible and hence $C^*(\epsilon, T) = \lambda_p(1-\epsilon)\gamma^{-1}(\epsilon)$. We approximate $\gamma^{-1}(\epsilon)$ for small ϵ by the Lagrange inversion theorem. Observe that $\gamma(\bar{c})$ is a smooth function of $\bar{c}$ and all derivatives exist. Expanding $\gamma^{-1}(\epsilon)$ around $\epsilon = 0$ by the Lagrange inversion theorem and using $\gamma(0) = 0$ yields

$$\begin{aligned} \gamma^{-1}(\epsilon) \quad &= \quad \sum_{n=1}^{\infty} \frac{d^{n-1}}{d\bar{c}^{n-1}} \left(\frac{\bar{c}}{\gamma(\bar{c})}\right)^n \Big|_{\bar{c}=0} \frac{\epsilon^n}{n!} \\ &= \quad \frac{\bar{c}\epsilon}{\gamma(\bar{c})}\,|_{\bar{c}=0} + o(\epsilon) \\ &\overset{(a)}{=} \quad \frac{\epsilon}{\lambda_p\beta_I + k} + o(\epsilon) \end{aligned} \tag{71}$$

where (a) follows by applying de L'Hôpital's rule. ■

We have the following observations

1) The constrained transmission capacity increases (slowly) with λ_p.

2) We also observe that the constrained transmission capacity for the cluster process is always less than that of a Poisson network (see Figure 8) and approaches $C_p(\epsilon, T)$ as $\lambda_p \to \infty$.
3) When FH-CDMA with intra-cluster frequency hopping is utilized, we have the cluster intensity $\bar{c}$ reduced by a factor M (spreading gain). One can easily obtain the constrained transmission capacity of this system to be
$$C^*_{FH}(\epsilon, T) = (1-\epsilon)\left(\frac{\epsilon \lambda_p M}{\lambda_p \beta_I + k} + o(\epsilon)\right)$$
When DS-CDMA is used, the constrained transmission capacity is $C^*_{DS}(\epsilon, T) = C^*(\epsilon, T/M)$. When the transmitters are spread as a Poisson point process, we have from [28], [29]
$$\ln\left(\frac{C_{FH}(\epsilon, T)}{C_{DS}(\epsilon, T)}\right) = (1 - 2/\alpha)\ln(M).$$
In Figure 9, we plot $\ln(C^*_{FH}(\epsilon, T)/C^*_{DS}(\epsilon, T))/\ln(M)$ with respect to spreading gain M, when the path loss function is $g(x) = \|x\|^{-\alpha}$ and $\epsilon = 0.01$. From the figure, we observe a similar $M^{1-2/\alpha}$ gain, even in the case of clustered transmitters.

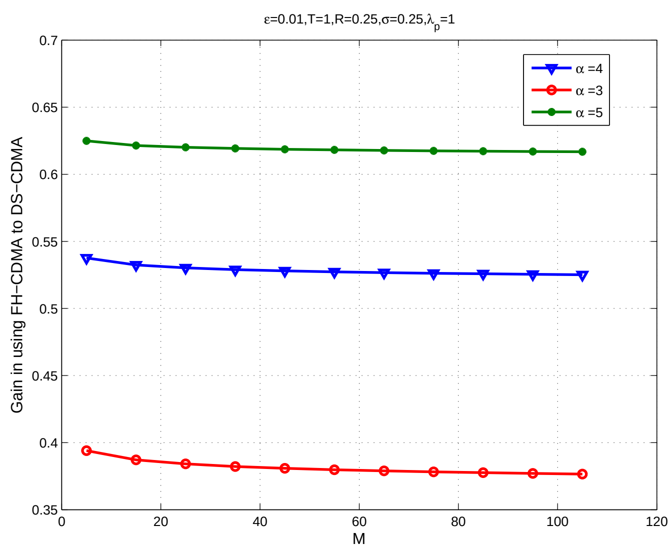

Fig. 9. $\ln(C^*_{FH}(\epsilon, T)/C^*_{DS}(\epsilon, T))/\ln(M)$ versus M for $\epsilon = 0.01, \lambda_p = 1$

V. Conclusions

Previous work characterizing interference, outage, and transmission capacity in large random networks exclusively focused on the homogeneous Poisson point process as the underlying node distribution. In this paper, we extend these results to clustered processes. Clustering may be geographical, *i.e.*, given by the spatial distribution of the nodes, or it may be induced logically by the MAC scheme. We use tools from stochastic geometry and Palm probabilities to obtain the conditional Laplace transform of the interference. Upper and lower bounds are obtained for the CCDF of the interference, for any stationary distribution of nodes and fading. We have shown that the distribution of interference depends heavily on the path-loss model considered. In particular, the existence of a singularity in the model greatly affects the results. This conditional Laplace transform is then used to obtain the probability of success in a clustered network with Rayleigh fading. We show clustering the transmitters is always beneficial for large link distances, while the clustering gain at smaller link distances depends on the path-loss model. The transmission capacity of clustered networks is equal to the one for homogeneous networks. However, care must be taken when defining this capacity since clustered processes have two parameters to optimize over. We also show that the transmission capacity of clustered network is equal to the Poisson distribution of nodes. We anticipate that the analytical techniques used in this work will be useful for other problems as well. In particular the conditional generating functionals are likely to find wide applicability.

Acknowledgments

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References

[1] E. S. Sousa and J. A. Silvester, "Optimum transmission ranges in a direct-sequence spread spectrum multihop packet radio network," *IEEE Journal on Selected Areas in Communications*, pp. 762–771, 1990.

[2] R. Mathar and J. Mattfeldt, "On the distribution of cumulated interference power in Rayleigh fading channels," *Wireless Networks*, vol. 1, no. 1, pp. 31–36, 1995.

[3] M. Westcott, "On the existence of a generalized shot-noise process," *Studies in Probability and Statistics. Papers in Honour of Edwin JG Pitman, North-Holland, Amsterdam*, p. 7388, 1976.

[4] S. Rice, "Mathematical analysis of random noise," *Selected Papers on Noise and Stochastic Processes*, pp. 133–294, 1954.

[5] J. Rice, "On Generalized Shot Noise," *Advances in Applied Probability*, vol. 9, no. 3, pp. 553–565, 1977.

[6] L. Heinrich and V. Schmidt, "Normal Convergence of Multidimensional Shot Noise and Rates of This Convergence," *Advances in Applied Probability*, vol. 17, no. 4, pp. 709–730, 1985.

[7] J. Ilow and D. Hatzinakos, "Analytic alpha-stable noise modeling in a Poisson field of interferers or scatterers," *Signal Processing, IEEE Transactions*, vol. 46, no. 6, pp. 1601–1611, 1998.

[8] M. Shao and C. Nikias, "Signal processing with fractional lower order moments: stable processes and their applications," *Proceedings of the IEEE*, vol. 81, no. 7, pp. 986–1010, 1993.

[9] F. Baccelli, B. Blaszczyszyn, and P. Muhlethaler, "An ALOHA protocol for multihop mobile wireless networks," *IEEE Transactions on Information Theory*, no. 2, Feb 2006.

[10] J. Venkataraman and M. Haenggi, "Optimizing the throughput in random wireless ad hoc networks," in *42st Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, Oct. 2004.

[11] J. Venkataraman, M. Haenggi, and O. Collins, "Shot noise models for the dual problems of cooperative coverage and outage in random networks," in *44st Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, Sept. 2006.

[12] S. Weber and J. G. Andrews, "A stochastic geometry approach to wideband ad hoc networks with channel variations," in *Proceedings of the Second Workshop on Spatial Stochastic Models for Wireless Networks (SPASWIN)*, Boston, MA, April 2006.

[13] M. Haenggi, "Outage and throughput bounds for stochastic wireless networks," *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on*, pp. 2070–2074, 2005.

[14] S. Weber, X. Yang, J. Andrews, and G. de Veciana, "Transmission capacity of wireless ad hoc networks with outage constraints," *Information Theory, IEEE Transactions on*, vol. 51, no. 12, pp. 4091–4102, 2005.

[15] S. Weber and J. G. Andrews, "Bounds on the SIR distribution for a class of channel models in ad hoc networks," in *Proceedings of the 49th Annual IEEE Globecom Conference*, San Francisco, CA, November 2006.

[16] D. Stoyan, W. S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*, 2nd ed., ser. Wiley series in probability and mathematical statistics. New York: Wiley, 1995.

[17] O. Kallenberg, *Random Measures*. Akademie-Verlag, Berlin, 1983.

[18] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*, 2nd ed. New York: Springer, 1998.

[19] D. R. Cox and V. Isham, *Point Processes*. London and New York: Chapman and Hall, 1980.

[20] L. Heinrich, "Asymptotic behaviour of an emperical nearest-neighbour distance function for stationary poisson cluster processes," *Math. Nachr.*, vol. 136, pp. 131–148, 1988.

[21] M. Baudin, "Likelihood and Nearest-Neighbor Distance Properties of Multidimensional Poisson Cluster Processes," *Journal of Applied Probability*, vol. 18, no. 4, pp. 879–888, 1981.

[22] K. H. Hanisch, "Reduction of the n-th moment measures and the special case of the third moment measure of stationary and isotropic planar point process," *Mathematische Operationsforschung and Statistik Series Statistics*, vol. 14, pp. 421–435, 1983.

[23] M. Westcott, "The probability generating functional," *Journal of Australian Mathematical Society*, vol. 14, pp. 448–466, 1972.

[24] S. Lowen and M. Teich, "Power-law shot noise," *Information Theory, IEEE Transactions on*, vol. 36, no. 6, pp. 1302–1318, 1990.

[25] G. Samorodnitsky, "Tail behavior of shot noise processes." [Online]. Available: http://citeseer.ist.psu.edu/603758.html

[26] H. Inaltekin and S. Wicker, "The behavior of unbounded path-loss models and the effect of singularity on computed network interference." *Fourth Annual IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks SECON 2007*, June 2007.

[27] G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*, 2nd ed. Wiley, 1999.

[28] J. Andrews, S. Weber, and M. Haenggi, "Ad hoc networks: To spread or not to spread," *To appear: IEEE Communications Magazine*, 2007.

[29] S. Weber, J. Andrews, X. Yang, and G. de Veciana, "Transmission capacity of wireless ad hoc networks with successive interference cancellation," *IEEE Transactions on Information Theory, under revision, submitted in*, 2005.

Appendix

A. *Proof of Lemma 3:*

Proof: We first evaluate the asymptotic behavior of $\mathcal{G}\left(F_h\left(\frac{y}{g(.-z)}\right)\right)$. Let $v(x) = F_h\left(\frac{y}{g(x-z)}\right)$. We have

$$\begin{aligned} M\left(\int_{\mathbb{R}^2} v(x+u)f(u)\mathrm{d}u\right) &= \exp\left(-\bar{c}\int_{\mathbb{R}^2}[1-F_h\left(\frac{y}{g(x+u-z)}\right)]f(u)\mathrm{d}u\right) \\ &\overset{(a)}{\sim} 1-\bar{c}\int_{\mathbb{R}^2}\left[1-F_h\left(\frac{y}{g(x+u-z)}\right)\right]f(u)\mathrm{d}u \end{aligned} \tag{72}$$

where (a) follows from the fact that $\exp(-x) = 1 - x + \mathcal{O}(x^2)$ for x close to 0 and $(1 - F_h) \to 0$ for large y. By a similar expansion of exp, (72) and the dominated convergence theorem, we have

$$\begin{aligned} &\exp\left(-\lambda_p\int_{\mathbb{R}^2} 1 - M(\int_{\mathbb{R}^2} v(x+u)f(u)\mathrm{d}u)\mathrm{d}x\right) \\ \sim\ & 1-\lambda_p\bar{c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[1-F_h\left(\frac{y}{g(x+u-z)}\right)]f(u)\mathrm{d}u\mathrm{d}x \\ =\ & 1-y^{-2/\alpha}\lambda_p\bar{c}\int_{\mathbb{R}^2}[1-F_h(\|x\|^\alpha)]\mathrm{d}x \end{aligned} \tag{73}$$

By change of variables and using $\lim_{y\to\infty}[1-F_h(y)]y^{2/\alpha}=0$ [27, p.198], we have

$$\int_{\mathbb{R}^2}[1-F_h(\|x\|^\alpha)]\mathrm{d}x=\pi\int_0^\infty \nu^{2/\alpha}\mathrm{d}F_h(\nu)$$

We similarly have

$$\begin{aligned}
&\int_{\mathbb{R}^2} M\left(\int_{\mathbb{R}^2} v(x+u)f(u)\mathrm{d}u\right)f(x)\mathrm{d}x \\
\sim\ & 1-\bar{c}\int_{\mathbb{R}^2}\left(1-\int_{\mathbb{R}^2}F_h\left(\frac{y}{g(x+u-z)}\right)f(u)\mathrm{d}u\right)f(x)\mathrm{d}x \\
=\ & 1-\bar{c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[1-F_h\left(y\|x+u-z\|^\alpha\right)\right]f(u)f(x)\mathrm{d}u\mathrm{d}x \\
=\ & 1-\bar{c}\int_{\mathbb{R}^2}\left[1-F_h\left(y\|x\|^\alpha\right)\right](f*f)(x+z)\mathrm{d}x \\
=\ & 1-\bar{c}y^{-2/\alpha}\int_{\mathbb{R}^2}\left[1-F_h\left(\|x\|^\alpha\right)\right](f*f)\left(\frac{x}{y^{1/\alpha}}+z\right)\mathrm{d}x \\
\overset{(a)}{\sim}\ & 1-\bar{c}(f*f)(z)y^{-2/\alpha}\int_{\mathbb{R}^2}\left[1-F_h\left(\|x\|^\alpha\right)\right]\mathrm{d}x
\end{aligned} \tag{74}$$

where (a) follows from the Lebesgue dominated convergence theorem ($f*f$ is a very nice function since f is a PDF). So we have $\mathcal{G}\left(F_h\left(\frac{y}{g(.-z)}\right)\right)\sim\theta_1 y^{-2/\alpha}$. For a Neyman-Scott cluster process, the second order product density is given by [16, p.158],

$$\rho^{(2)}(r)=\lambda^2+\frac{\lambda\mu(r)}{\pi\bar{c}}\sum_{n=2}^{\infty}p_n n(n-1)$$

where p_n is the distribution of the number of points in the representative cluster. $\mu(r)/\pi$ denotes the density of the distribution function for the distance between two independent random points which were scattered using the distribution $f(x)$ of the representative cluster. When the number of points inside each cluster is Poisson distributed with mean $\bar{c}$, we have $\sum_{n=2}^{\infty}p_n n(n-1)=\bar{c}^2$. We also have $\mu(x)/\pi=(f*f)(x)$ Estimating $\varphi(y)$ we have

$$\begin{aligned}
\varphi(y) &= \frac{1}{y\lambda}\int_{\mathbb{R}^2}g(x-z)\rho^{(2)}(x)\int_0^{y/g(x-z)}\nu\mathrm{d}F_h(\nu)\mathrm{d}x \\
&= \underbrace{\frac{\lambda}{y}\int_{\mathbb{R}^2}\|x\|^{-\alpha}\int_0^{y\|x\|^\alpha}\nu\mathrm{d}F_h(\nu)d}_{T_1} \\
&\quad+\underbrace{\frac{\bar{c}}{y\pi}\int_{\mathbb{R}^2}g(x-z)\mu(x)\int_0^{y/g(x-z)}\nu\mathrm{d}F_h(\nu)\mathrm{d}x}_{T_2}
\end{aligned} \tag{75}$$

By change of variables, we have

$$T_1 = \frac{2\pi\lambda y^{-2/\alpha}}{\alpha-2}\int_0^\infty \nu^{2/\alpha}\mathrm{d}F_h(\nu) \tag{76}$$

For the term T_2,

$$\begin{aligned} T_2 &= \frac{\bar{c}}{y\pi}\int_0^\infty \nu \mathrm{d}F_h(\nu)\int_{\mathbb{R}^2}\|x-z\|^{-\alpha}1_{\|x-z\|^\alpha>vy^{-1}}\mu(x)\mathrm{d}x \\ &= \frac{\bar{c}}{\pi y^{2/\alpha}}\int_0^\infty \nu \mathrm{d}F_h(\nu)\int_{\mathbb{R}^2}\|x\|^{-\alpha}1_{\|x\|^\alpha>v}\mu\left(\frac{x}{y^{1/\alpha}}+z\right)\mathrm{d}x \\ &\sim \frac{\mu(z)\bar{c}}{\pi y^{2/\alpha}}\int_0^\infty \nu \mathrm{d}F_h(\nu)\int_{\mathbb{R}^2}\|x\|^{-\alpha}1_{\|x\|^\alpha>v}\mathrm{d}x \\ &= \frac{y^{-2/\alpha}2\mu(z)\bar{c}}{\alpha-2}\int_0^\infty \nu^{2/\alpha}\mathrm{d}F_h(\nu) \end{aligned} \tag{77}$$

So we have $\varphi(y)\sim\theta_2 y^{-2/\alpha}$. Hence from Theorem 1, we have $\bar{F}_I^l(y)\sim\theta_1 y^{-2/\alpha}$and $\bar{F}_I^u(y)\sim(\theta_1+\theta_2)y^{-2/\alpha}$.

■

B. Outage probability, in Poisson cluster process when the number of cluster points are fixed.

In this subsection we derive the conditional Laplace transform in a Poisson cluster process, when the number of points in each cluster are fixed to be $\bar{c}\in\mathbb{N}$ and $\bar{c}>0$. We also assume that each point is independently distributed with density $f(x)$. In this case the moment generating function of the number of points in the representative cluster is given by

$$M(z) = z^{\bar{c}}$$

Using the same notation as in Section II-B, and from (11) and (13), we have

$$\begin{aligned} E_0^!\left(\prod_{x\in\phi}v(x)\right) &= \tilde{G}(v)\int_N\prod_{x\in\psi}v(x)\tilde{\Omega}_0^!(d\psi) \\ &= \tilde{G}(v)\int_N\prod_{x\in\psi}v(x)\int_{\mathbb{R}^2}\Omega_y^!(d\psi_y)f(y)\mathrm{d}y \\ &= \tilde{G}(v)\int_{\mathbb{R}^2}\int_N\prod_{x\in\psi}v(x)\Omega_Y^!(d\psi_y)f(y)\mathrm{d}y \\ &\overset{(a)}{=} \tilde{G}(v)\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}v(x-y)f(x)\right)^{\bar{c}-1}f(y)\mathrm{d}y \end{aligned} \tag{78}$$

where (a) follows from the fact that the points are independently distributed and we are not counting the point at the origin. In this case $\tilde{G}(v)$ is given by

$$\tilde{G}(v) = \exp\left\{-\lambda_p\int_{\mathbb{R}^2}1-\left(\int_{\mathbb{R}^2}v(x+y)f(y)\mathrm{d}y\right)^{\bar{c}}\mathrm{d}x\right\}$$

Hence the success probability (Rayleigh fading) is given by

$$\mathbb{P}(\text{success}) = \exp\left\{-\lambda_p\int_{\mathbb{R}^2}1-\tilde{\beta}(R,y)^{\bar{c}}\mathrm{d}y\right\}\int_{\mathbb{R}^2}\tilde{\beta}(R,y)^{\bar{c}-1}f(y)\mathrm{d}y \tag{79}$$

where

$$\tilde{\beta}(R,y) = \int_{\mathbb{R}^2}\frac{f(x)}{1+\frac{g(R)}{T}g(x-y-z)}\mathrm{d}x$$

C. Outage probability of Nakagami-m fading

Here, we derive the success probability when the fading distribution is Nakagami-m distributed. We also assume $m \in \mathbb{N}$ and $W = 0$. The PDF of the power fading coefficient $y = h_x$ is given by

$$p(y) = \frac{1}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m y^{m-1} e^{-my/\Omega}$$

From (35), we have

$$\mathbb{P}(\text{success}) = \mathbb{P}\left(\frac{hg(z)}{W + I_{\phi\setminus\{0\}}(z)} \geq T\right)$$

$$\mathbb{P}(\text{success}) = 1 - \left(\frac{T}{g(z)}\right)^m \int_0^\infty \frac{1}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m y^{m-1} e^{-\frac{Tm}{\Omega g(z)} y} \mathbb{P}^{!0}(I_\phi(z) > y)\mathrm{d}y \tag{80}$$

Using integration by parts we get,

$$\begin{aligned} \mathbb{P}(\text{success}) &= \frac{1}{\Gamma(m)} \int_0^\infty \Gamma(m, \frac{Tm}{\Omega g(z)} y) d\mathbb{P}^{!0}(I_\phi(z) \leq y) \\ &\stackrel{(a)}{=} \frac{(m-1)!}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{1}{k!} \int_0^\infty e^{-\frac{Tm}{\Omega g(z)} y} y^k d\mathbb{P}^{!0}(I_\phi(z) \leq y) \\ &\stackrel{(b)}{=} \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} \frac{d^k}{ds^k} \mathcal{L}_{I_\phi(z)}(s)|_{s=Tm/\Omega g(z)} \end{aligned} \tag{81}$$

where (a) follows from the series expansion of incomplete Gamma function when m is an integer and (b) follows from the properties of the Laplace transform and $\Gamma(m) = (m-1)!$ when m is an integer. We also have

$$\mathcal{L}_{h_x}(sg(x-z)) = \frac{1}{(1 + \frac{\Omega}{m} sg(x-z))^m}$$

Hence from Lemma 2, we have

$$\mathcal{L}_{I_\phi(z)}(s) = \exp\left[-\lambda_p \int_{\mathbb{R}^2} 1 - \exp(-\bar{c}\bar{\beta}(s,z,y))\mathrm{d}y\right] \int_{\mathbb{R}^2} \exp(-\bar{c}\bar{\beta}(s,z,y)) f(y)\mathrm{d}y \tag{82}$$

where

$$\bar{\beta}(s,z,y) = 1 - \int_{\mathbb{R}^2} \frac{1}{(1 + \frac{\Omega}{m} sg(x-y-z))^m} f(x)\mathrm{d}x$$

For integer $m \geq 1$, $\mathbb{P}(\text{success})$ can be evaluated from (81) and (82). For $m = 1$, the probability evaluated from (81) and (82) matches that of Lemma 4.

D. Proof of Lemma 7

Proof:

$$\begin{aligned}\frac{P_p(\lambda_p\bar{c})}{P_1(R,\bar{c},\lambda_p)} &= \exp\Big[-\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,y)\mathrm{d}y + \lambda_p\int_{\mathbb{R}^2}(1-\exp[-\bar{c}\beta(R,y)])\mathrm{d}y\Big] \\ &= \exp\Big[-\lambda_p\int_{\mathbb{R}^2}\{\underbrace{\bar{c}\beta(R,y) - 1 + \exp\Big[-\bar{c}\beta(R,y)\Big]}_{\nu(R,y)}\}\mathrm{d}y\Big] \quad (83)\end{aligned}$$

Since $1-\exp(-ax) \le ax$, we have that $\nu(R,y) > 0$. We also have from the dominated convergence theorem and (1)

$$\lim_{R\to\infty}\nu(R,y) = \frac{\bar{c}}{1+T^{-1}} - 1 + \exp\left(-\frac{\bar{c}}{1+T^{-1}}\right) > 0$$

which is a constant. So using Fatou's lemma [27] ($\liminf\int f_n \ge \int(\liminf f_n),\ f_n > 0$), we have

$$\begin{aligned}\lim_{R\to\infty}\frac{P_p(\lambda_p\bar{c})}{P_1(R,\bar{c},\lambda_p)} &= \exp[-\lambda_p\lim_{R\to\infty}\int_{\mathbb{R}^2}\nu(R,y)\mathrm{d}y] \\ &\le \exp[-\lambda_p\int_{\mathbb{R}^2}\lim_{R\to\infty}\nu(R,y)\mathrm{d}y] \\ &= \exp[-\lambda_p\infty] = 0 \quad (84)\end{aligned}$$

■

E. Proof of Lemma 8

Proof: From (55), the probability of success is

$$\mathbb{P}(\text{success}) = P_p(\lambda_p\bar{c})\underbrace{\exp\Big[\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,y)\eta(\bar{c},R,y)\mathrm{d}y\Big]}_{T_1}\underbrace{P_2(R,\bar{c})}_{T_2} \quad (85)$$

where $\eta(\bar{c},R,y) = \nu(\bar{c}\beta(R,y))$ and $\nu(x) = (\exp(-x)-1+x)/x$ an increasing function of x. From Young's inequality [27, Sec. 8.7] we have $\beta(R,y) \le \min\{1, \sup\{f(x)\}R^2T^{2/\alpha}C(\alpha)\}$. Hence

$$\eta(\bar{c},R,y) \le \nu(\bar{c}\min\{1,\sup\{f(x)\}R^2T^{2/\alpha}C(\alpha)\})$$

With a slight abuse of notation, let $\eta(\bar{c},R) = \nu(\bar{c}\min\{\sup\{f(x)\}R^2T^{2/\alpha}C(\alpha),1\})$. Hence

$$\begin{aligned}T_1 &\le \exp[\lambda_p\bar{c}\int_{\mathbb{R}^2}\beta(R,y)\eta(\bar{c},R)\mathrm{d}y] \\ &= \exp[\lambda_p\bar{c}T^{2/\alpha}R^2\eta(\bar{c},R)C(\alpha)] \quad (86)\end{aligned}$$

Also observe that $\eta(\bar{c},R) \lessapprox R^2$. So $T_1 \lessapprox \exp(R^4)$.

$$\begin{aligned}T_2 &= \int_{\mathbb{R}^2}1-\bar{c}\beta(R,y)+\bar{c}\beta(R,y)\sum_{k=2}^{\infty}\frac{(-1)^n}{n!}(\bar{c}\beta(R,y))^{n-1}f(y)\mathrm{d}y \\ &\le \int_{\mathbb{R}^2}[1-\bar{c}\beta(R,y)+\bar{c}\beta(R,y)\eta(\bar{c},R)]f(y)\mathrm{d}y \\ &= 1-[1-\eta(\bar{c},R)]\bar{c}\int_{\mathbb{R}^2}\beta(R,y)f(y)\mathrm{d}y \quad (87)\end{aligned}$$

If one considers x and y as identical and independent random variables with density functions f, we then have $\int_{\mathbb{R}^2}\beta(R,y)f(y)\mathrm{d}y = E[\frac{1}{1+\frac{g(R)}{T}\|x-y-R\|^{\alpha}}]$. Let $0<\kappa<1$ be some constant. Using the Chebyshev inequality we get

$$\begin{aligned}
E\left[\frac{1}{1+\frac{g(R)}{T}\|x-y-R\|^{\alpha}}\right] &\geq \kappa P\left[\frac{1}{1+\frac{g(R)}{T}\|x-y-R\|^{\alpha}}\geq\kappa\right] \\
&= \kappa P\Big[\|x-y-R\|\leq(\frac{1}{\kappa}-1)^{1/\alpha}RT^{1/\alpha}\Big] \quad (**) \qquad (88)
\end{aligned}$$

The PDF of $z=x-y$ is given by $(f*f)(z)$, since y is rotation-invariant. Choosing $\kappa=T/(1+T)$ we have

$$\begin{aligned}
(**) &= \frac{T}{1+T}\int_{B(R,R)}(f*f)(x)\mathrm{d}x \\
&\geq \frac{T}{1+T}\int_{B(R,R)}\|x\|\mathrm{d}x \\
&= R^3\underbrace{\frac{T}{1+T}\int_{B(1,1)}\|x\|\mathrm{d}x}_{C_2} \qquad (89)
\end{aligned}$$

So we have

$$\begin{aligned}
P_2 &\leq 1-[1-\eta(\bar{c},R)]R^3C_2 \\
&\lessapprox 1-R^3+R^5 \qquad (90)
\end{aligned}$$

Also we have $T_1\lessapprox\exp(R^4)\lessapprox 1+1.01R^4$. So we have $P_2T_1\lessapprox 1-R^3+R^5-1.01R^7+1.01R^9<1$ for small $R\neq 0$. *Hence for small* R *we have* $\mathbb{P}(success)\leq P_p(\lambda_p\bar{c})$.

■

F. Proof of Lemma 11

Proof: We find $C_u(\epsilon,T)$ and hence upper bound the transmission capacity. We have from the derivation of Lemma 8

$$P(\lambda,T) \leq P_p(\lambda_p\bar{c})\exp[\lambda_p\bar{c}\beta_I\eta(\bar{c},R)]P_2(R,\bar{c}) = P_u(\bar{c}\lambda_p,T) \qquad (91)$$

where $\eta(\bar{c},R)=(\exp(-\bar{c}\hat{\beta})-1+\bar{c}\hat{\beta})/\bar{c}\hat{\beta}$. With $A_u=\{\lambda_p\bar{c},\ P_u(\lambda_p\bar{c},T)\geq 1-\epsilon\}$, it is sufficient to prove $\sup A_u\leq C_p(\epsilon,T)$. Also observe that $P_u(\bar{c}\lambda_p,T)\to 0$ as $\bar{c}\to\infty$ independent of λ_p. Hence we can assume $\bar{c}$ is finite for the proof. We proceed by contradiction.

Let $\sup A_u > C_p(\epsilon,T)$. Hence there exists a $\delta>0,\ \lambda_p\geq 0,\ \bar{c}\geq 0$ such that $\lambda_p\bar{c}=(C_p(\epsilon,T)/(1-\epsilon))+\delta\in A_u$. At this value of $\lambda_p\bar{c}$ we have

$$\begin{aligned}
1-\epsilon\leq P_u(\bar{c}\lambda_p,T) &= (1-\epsilon)P_p(\delta)\exp[\eta(\bar{c},R)\{\ln(\frac{1}{1-\epsilon})+\delta\beta_I\}]P_2(R,\bar{c}) \\
&= (1-\epsilon)^{1-\eta(\bar{c},R)}\underbrace{P_p(\delta(1-\eta(\bar{c},R)))}_{T_1}P_2(R,\bar{c}) \qquad (92)
\end{aligned}$$

From the derivation of Lemma 8, we have $P_2(R,\bar{c}) \leq 1 - [1-\eta(\bar{c},R)]\bar{c}k$, with equality only when $\bar{c}=0$. Hence we have

$$p_u(\bar{c}\lambda_p, T) \quad \leq \quad T_1(1-\epsilon)^{1-\eta(\bar{c},R)}(1-[1-\eta(\bar{c},R)]\bar{c}k) \tag{93}$$

Since $\frac{\exp(-x)-(1-x)}{x} \leq \frac{x}{1+x}$, we have $\eta(\bar{c},R) \leq \bar{c}\hat{\beta}/(1+\bar{c}\hat{\beta})$. Using the upper bound for $\eta(\bar{c},R)$

$$\begin{aligned} p_u(\bar{c}\lambda_p, T) &\leq T_1(1-\epsilon)(1-\epsilon)^{-\frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}}\left(1-\left[1-\frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}\right]\bar{c}\hat{\beta}\rho(T)\right) \\ &= T_1(1-\epsilon)\underbrace{(1-\epsilon)^{-\frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}}\left(1-\frac{\bar{c}\hat{\beta}\rho(T)}{1+\bar{c}\hat{\beta}}\right)}_{T_2} \end{aligned} \tag{94}$$

Using the inequality $1-ay \leq (1-b)^y,\ b \leq 1-e^{-a}, y \geq 0$, substituting $y = \frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}},\ b=\epsilon, a=\rho(T)$, we get $T_2 \leq 1$. Hence we have

$$p_u(\bar{c}\lambda_p, T) \quad \leq \quad (1-\epsilon)P_p(\delta(1-\eta(\bar{c},R))) \tag{95}$$

So if $\delta > 0$, and $\bar{c}$ finite, we also have $P_p(\delta(1-\eta(\bar{c},R))) < 1$. So we have a contradiction from (92) and (95). Hence there exists no such δ and hence $\sup A_u \leq C_p(\epsilon,T)$. We can achieve $C_u(\epsilon,T) = C_p(\epsilon,T)$, by using $\lambda_p = n\frac{C_p(\epsilon,T)}{1-\epsilon} - 1, \bar{c} = 1/n$ for n very large. As $n \to \infty, P_u(\bar{c}\lambda_p, T) \to P_p(\bar{c}\lambda_p, T)$. ■