

# Transmission Capacity of Ad Hoc Networks with Spatial Diversity

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**Abstract**—This paper derives the outage probability and transmission capacity of ad hoc wireless networks with nodes employing multiple antenna diversity techniques, for a general class of signal distributions. This analysis allows system performance to be quantified for fading or non-fading environments. The transmission capacity is given for interference-limited uniformly random networks on the entire plane with path loss exponent  $\alpha > 2$  in which nodes use: (1) static beamforming through  $M$  sectorized antennas, for which the increase in transmission capacity is shown to be  $\Theta(M^2)$  if the antennas are without sidelobes, but less in the event of a nonzero sidelobe level; (2) dynamic eigenbeamforming (maximal ratio transmission/combining), in which the increase is shown to be  $\Theta(M^{\frac{2}{\alpha}})$ ; (3) various transmit antenna selection and receive antenna selection combining schemes, which give appreciable but rapidly diminishing gains; and (4) orthogonal space-time block coding, for which there is only a small gain due to channel hardening, equivalent to Nakagami- $m$  fading for increasing  $m$ . It is concluded that in ad hoc networks, static and dynamic beamforming perform best, selection combining performs well but with rapidly diminishing returns with added antennas, and that space-time block coding offers only marginal gains.

**Index Terms**—Ad hoc networks, multiple antennas, stochastic geometry, outage capacity.

## I. INTRODUCTION

PRIOR work on ad hoc network capacity has focused on the limiting behavior as the network grows large. For the purpose of ascertaining the effect that multiple antennas have on the capacity of the network, the more pertinent question is how the capacity scales with the number of antennas at each node. Naturally, this scaling will differ depending on the way that the antennas are utilized. The goal of this paper is to determine which multiple-antenna techniques perform best in a given network density or similarly, which technique can support the densest network.

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Multi-antenna systems (MIMO) are currently of great interest in all wireless communication systems due to their potential to combat fading, increase spectral efficiency, and reduce interference. Over the past decade, many different MIMO techniques have been proposed, which can be grouped into three broad categories: diversity-achieving, beam-steering, and spatial multiplexing. Diversity-achieving techniques increase reliability by combatting or exploiting channel variations. Beam-steering techniques increase received signal quality by focusing desired energy or attenuating undesired interference. Spatial multiplexing aggressively increases the data rate by transmitting independent data symbols across the antenna array. In this paper we focus on the first two types of techniques, which do not increase the number of independent datastreams and hence are easier to fairly compare. We also expect that these techniques will be more relevant than spatial multiplexing in interference-limited ad hoc networks since sending a single datastream in low-SNR links is superior in terms of both performance and implementation complexity [1], [2]. This paper develops a framework for comparing the utility of the diversity-providing and beam-steering MIMO techniques, with the goal of providing insight on how to use multiple antennas in ad hoc networks.

### A. Background and Related Work

Recent advances in characterizing network capacity were sparked by [3] with its notion of transport capacity and a number of works have followed in the same vein including [4], [5], and [6]. These studies focus on the behavior of end-to-end network capacity in the limit as the number of nodes grows large under a variety of models of node interaction and fading conditions. These confirm the basic intuition from [3] that, under traditional technological or physical limitations on node cooperation and signal reception, transmissions require “area” in which to take place and so per node end-to-end throughput decays as  $\Theta(\frac{1}{\sqrt{n}})$  for  $n$  nodes in the network. A fundamental change occurs with significant mobility as [7] and [8] show since optimal routing can take on new forms to tradeoff throughput and delay.

An alternative characterization of ad hoc network capacity was developed in [9] which defined rate regions for given network configurations and traffic needs. This was extended to the MIMO case in [10] and the notion of “capacity region” was extended in several ways. This versatile approach has the drawback of being prohibitively computationally intensive for analyzing large networks. It also focuses on large network

optimization problems that would be difficult to solve in a distributed system at present.

A straightforward way to evaluate a physical layer technique under per node service requirements is to determine the maximum density of concurrent transmissions, or the *optimal contention density*, for which each node's requirements are still met. This leads naturally to the transmission capacity metric which is defined in [11] to be the maximum allowable spatial density of successful transmissions multiplied by their data rate given an outage constraint. For an outage constraint  $\epsilon$  and a transmission data rate  $b$  in bits/s/Hz or per channel use, the transmission capacity is given by  $c_\epsilon = b(1 - \epsilon)\lambda_\epsilon$  for the optimal contention density  $\lambda_\epsilon$ . The transmission capacity is then the area spectral efficiency resulting from the optimal contention density.

Computing the transmission capacity is made possible by using a spatial point process to model node positions, as pioneered in the analysis of wireless networks by [12]. More recently, Haenggi, *et al.* in [13], [14], and [15] emphasized the importance of network topology by characterizing some of the distinctions in throughput, interference, and outage in regular as well as clustered random networks. This approach was also taken in [16] which developed bounds on the transmission capacity for general fading models as well as power control and scheduling schemes with only individual channel state information and single antennas.

Several papers, including [17], [18], [19], have studied the effects of cochannel interference on MIMO. However, these studies lack a clear link between point-to-point throughput and network performance gains. It is presently unclear which MIMO technologies yield the highest gains in large random networks. For example, [20] uses a game-theoretic analysis to show that capacity is maximized for mutually interfering sources when each sends only one datastream, while [21] and [22] suggest capacity is improved through spatially multiplexing potentially multiple transmissions; however, [21] again focuses on asymptotics in the number of nodes and the results of [22] are obscured by the mobility/delay issue.

## B. Contributions

This paper analyzes networks with single-datastream MIMO diversity techniques including beamforming, antenna sectorization, space-time block coding, and selection combining in Rayleigh fading, terms which we will make precise in the course of the paper. The gains in transmission capacity of each are shown and compared, especially as a function of the number of antennas. Results are also given for Nakagami- $m$  fading for integer  $m$  to compare methods in line-of-sight versus non-line-of-sight propagation environments and to assist in interpreting channel hardening gains. While spatial multiplexing techniques are omitted, they are left as future work though some of the results developed here will also be applicable to spatial multiplexing systems. Since diversity techniques are robust in noise-limited environments and generally reasonable to implement, they constitute an important subset of the primary MIMO techniques. Also, as indicated in [20] a game-theoretic analysis indicates optimality of single stream techniques in interference limited environments.

The goal of the paper will be to establish several clear relations between the optimal contention density and the number of antennas, which we list below. More precisely, for random wireless networks on the entire plane using the above MIMO techniques in block fading channels with path loss, this paper determines transmission capacity, the scaling of the optimal contention density with the number of antennas, and outage probabilities as a function of network parameters. Under small outage constraints, for  $M_t$  and  $M_r$  the number of transmit and receive antennas, respectively, and  $\alpha > 2$  is the path loss exponent, we have:

- 1) Ideal Sectorized Antennas:  $\lambda_\epsilon = \Theta(M_t M_r)$
- 2) Sectorized Antennas with sidelobe level  $\gamma \in [0, 1]$ :  $\lambda_\epsilon = \Theta \left[ \left( \frac{M}{1 + \gamma \frac{2}{\alpha}(M-1)} \right)^2 \right]$  for  $M = M_t = M_r$ .
- 3) Maximal Ratio Combining (MRC):  $\lambda_\epsilon = \Theta(M_r^{\frac{2}{\alpha}})$
- 4) Maximal Ratio Transmission (MRT) and Combining:  $\lambda_\epsilon = O((M_t M_r)^{\frac{2}{\alpha}})$ ,  $\lambda_\epsilon = \Omega(\max\{M_t, M_r\}^{\frac{2}{\alpha}})$
- 5) Orthogonal Space-Time Block Coding (OSTBC):  $\lambda_\epsilon = \Theta(M_r^{\frac{2}{\alpha}})$ .

The orderwise results demonstrate that as spatial diversity techniques increase the SINR, network throughput increases better than logarithmically in the number of antennas. In particular, these relations demonstrate that beamforming, either static or dynamic, achieves the most network transmission capacity increase among diversity techniques. On the other hand, space-time block coding yields little, especially for more antennas than two. The results also highlight the advantages of achieving diversity at the receiver since open loop transmit diversity techniques have some specific drawbacks to be discussed and receiver techniques do not require feedback.

The remainder of the paper is organized as follows: Section II presents the network model and derives properties for Poisson shot noise functionals applicable to large class of MIMO techniques. Section III discusses the optimal contention density for single antenna systems in Nakagami- $m$  fading. The optimal contention density for networks of nodes with multiple sectorized antennas is derived in Section IV. Section V derives the outage probabilities and optimal contention densities for receive MRC systems and Section V-B does the same for MIMO MRT/MRC systems. Networks using OSTBCs are analyzed in Section VI. Section VII gives transmission capacity results for selection combining in ad hoc networks and Section VIII concludes.

## II. THE NETWORK MODEL AND ANALYTICAL METHODS

### A. The Model

This section defines the network model and presents some results on Laplace functionals of Poisson shot noise processes which will be used in later sections. In order to focus on the physical layer, consider a wireless network operating a random access protocol in the style of slotted ALOHA without power control. As discussed in [11], this model includes the collision behavior of practical distributed systems while neither addressing nor precluding the issue of routing. It

also provides a way to give a clear relationship between the network throughput and the number of antennas employed for each technique. Let the distribution of transmitting nodes in the network be a stationary marked Poisson point process with intensity  $\lambda$  in  $\mathbb{R}^2$ ; the process is denoted by  $\Phi$ . To analyze the performance of a random access wireless network, consider a typical receiver located at the origin. As a result of Palm probabilities of a Poisson process, conditioning on the event of a node lying at the origin does not affect the statistics of the rest of the process (see [23], ch. 2). Moreover, due to stationarity of the Poisson process, the statistics of signal reception at this receiver are seen by any receiver.

To model propagation through the wireless channel, let signals be subject to path loss attenuation model  $d^{-\alpha}$  for a distance  $d$  with exponent  $\alpha > 2$  as well as small scale fading for either a Rayleigh or Nakagami- $m$  fading distribution with unit mean. Also, let all nodes transmit with the same power  $\rho$ . For such a channel, the typical receiver obtains desired signal power  $\rho S_0 R^{-\alpha}$  for some fixed transmitter-receiver separation distance  $R$ , and with a fading power factor  $S_0$  on the signal from its intended transmitter, labeled 0. The interfering nodes, numbered 1, 2, 3, ... constitute the marked process  $\Phi = \{(X_i, S_i)\}$ , with  $X_i$  denoting the location of the  $i$ th transmitting node, and with marks  $S_i$  that denote fading factors on the power transmitted from the  $i$ th node and then received by the typical receiver. Thus the receiver receives interference power  $\rho S_i |X_i|^{-\alpha}$  from the  $i$ th interfering node ( $|\cdot|$  denoting magnitude). For single-antenna narrowband systems in Rayleigh fading channels, for example, the power factors  $S_0$  and  $S_i$  are distributed exponentially with unit mean so that the mean interfering power is governed by transmit power and path loss.

Successful transmission occurs if the inequality

$$\frac{\rho S_0 R^{-\alpha}}{\rho I_\Phi + N_0} \geq \beta \quad (1)$$

is satisfied for some target signal-to-interference-and-noise ratio (SINR)  $\beta$ , aggregate co-channel interference  $\rho I_\Phi$ , and thermal noise  $N_0$ . The aggregate interference is a Poisson shot noise process (scaled by  $\rho$ ), which is a sum over the marked point process:

$$I_\Phi = \sum_{X_i \in \Phi} S_i |X_i|^{-\alpha} \quad (2)$$

with  $|X_i|$  denoting the distance of  $X_i$  from the origin. From here on, it will be assumed that the network is interference limited, with  $\rho I_\Phi \gg N_0$  so that thermal noise is negligible. Following [24], the probability of successful transmission for a typical receiver is:

$$\begin{aligned} \mathbf{P}(SIR \geq \beta) &= \mathbf{P}\left(\frac{\rho S_0 R^{-\alpha}}{\rho I_\Phi} \geq \beta\right) = \mathbf{P}(S_0 \geq \beta R^\alpha I_\Phi) \\ &= \int_0^\infty \mathbf{P}(S_0 \geq s \beta R^\alpha) f_{I_\Phi}(s) ds \\ &= \int_0^\infty F_{S_0}^c(s \beta R^\alpha) f_{I_\Phi}(s) ds \end{aligned} \quad (3)$$

where the third step is reached by conditioning on  $s$  and  $F^c(\cdot)$  denotes a complementary cumulative distribution function (CCDF). In the single antenna (SISO) case, the received signal

power is exponentially distributed with  $F_{S_0}^c(s \beta R^\alpha) = e^{-s \beta R^\alpha}$  so that

$$\mathbf{P}(SIR \geq \beta) = \int_0^\infty e^{-s \beta R^\alpha} f_{I_\Phi}(s) ds. \quad (4)$$

This is now a Laplace transform of the PDF of  $I_\Phi$  which gives  $\mathbf{P}(SIR \geq \beta) = \mathcal{L}_{I_\Phi}(\beta R^\alpha)$ . The Laplace transform for a general Poisson shot noise process in  $\mathbb{R}^2$  with independent, identically distributed (i.i.d.) marks  $S_i$  is given by [25]

$$\mathcal{L}_{I_\Phi}(\zeta) = \exp\left\{-\lambda \int_{\mathbb{R}^2} 1 - E\left[e^{-\zeta S_i |x|^{-\alpha}}\right] dx\right\} \quad (5)$$

where the expectation, denoted by  $E[\cdot]$ , is over  $S$  which has the same distribution as any  $S_i$ . Note that we are using the simplified attenuation function  $|d|^{-\alpha}$ . While this model is inaccurate in the near field, most notably because it explodes at the origin, for systems operating primarily in the far field (e.g.,  $R$  is many carrier wavelengths), this inaccuracy has negligible effect for the purpose of calculating outage probabilities. One can modify the path loss function to  $\frac{1}{1+|d|^\alpha}$ , for example, as mentioned in [24] and perform the same analysis. The result is that for  $R$  well in the far field, this modification leads to the same transmission capacity conclusions though with more cumbersome analysis.

For Rayleigh fading channels, i.e.,  $S_0, S_i \sim \text{Exp}(1)$ , (5) simplifies to

$$\mathcal{L}_{I_\Phi}(\zeta) = \exp\left\{-2\pi\lambda \int_0^\infty \frac{u}{1+|u|^\alpha/\zeta} du\right\} = e^{-\lambda C \zeta^{\frac{2}{\alpha}}} \quad (6)$$

with  $\mathcal{L}_{I_\Phi}(\zeta)$  evaluated at  $\zeta = \beta R^\alpha$  and  $C = \frac{2\pi}{\alpha} \Gamma(\frac{2}{\alpha}) \Gamma(1 - \frac{2}{\alpha})$ , with  $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$  being the Gamma function. Note that in general  $C$  depends on  $S_i$ , and so for some systems, it may no longer be a function of the path loss exponent alone. For all cases considered in this paper, the integral in (5)

$$\int_{\mathbb{R}^2} 1 - E\left[e^{-\zeta S_i |x|^{-\alpha}}\right] dx$$

where  $\zeta$  is evaluated at  $\beta R^\alpha$ , will be proportional to  $(\beta R^\alpha)^{\frac{2}{\alpha}}$ . This has the simple sphere packing interpretation that each transmission takes up an ‘‘area’’ proportional to  $(\beta^{1/\alpha} R)^2$ .

## B. The Optimal Contention Density

Applying a small outage constraint (e.g.,  $\epsilon < .1$ ) to (4), the network just meets this constraint when

$$\mathbf{P}(SIR \geq \beta) = 1 - \epsilon = e^{-\lambda C R^2 \beta^{\frac{2}{\alpha}}} \quad (7)$$

and solving for  $\lambda$  yields the optimal contention density:

$$\lambda_\epsilon = \frac{-\ln(1 - \epsilon)}{C R^2 \beta^{\frac{2}{\alpha}}} = \frac{\epsilon}{C R^2 \beta^{\frac{2}{\alpha}}} + \Theta(\epsilon^2). \quad (8)$$

Since the results herein will focus on the small outage regime, it will be convenient to introduce the notation  $\lambda_\epsilon = \bar{\lambda}_\epsilon + O(\epsilon^2)$  allowing equations to be expressed in terms of  $\bar{\lambda}_\epsilon$  with the  $O(\epsilon^2)$  error terms merely implied. The result (8) given in [11] and [24] can be generalized through the following Theorem.

*Theorem 1:* Let the interfering transmitters form a Poisson process of intensity  $\lambda$  around a typical receiver with the outage probability being  $\mathbf{P}(SIR \leq \beta) = \mathbf{P}\left(\frac{\rho S_0 R^{-\alpha}}{\rho I_\Phi} \leq \beta\right)$  with fixed  $\rho$ ,  $\beta$ ,  $R$ , and  $\alpha$ . Suppose  $F_{S_0}^c$  takes the form

$$F_{S_0}^c(x) = \sum_{n \in \mathcal{N}} e^{-nx} \sum_{k \in \mathcal{K}} a_{nk} x^k \quad (9)$$

for finite sets<sup>1</sup>  $\mathcal{N}, \mathcal{K} \subset \mathbb{N}$ , and suppose  $S_0$  is independent of  $I_\Phi$ , then

$$\mathbf{P}(SIR \geq \beta) = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \left[ a_{nk} \left(-\frac{\zeta}{n}\right)^k \frac{d^k}{d\zeta^k} \mathcal{L}_{I_\Phi}(\zeta) \right]_{\zeta = n\beta R^\alpha} \quad (10)$$

Furthermore, for a small outage constraint  $\epsilon$ , the optimal contention density is given by:

$$\bar{\lambda}_\epsilon = \frac{K_\alpha}{C_\alpha} \frac{\epsilon}{R^2 \beta^{\frac{2}{\alpha}}} \quad (11)$$

for

$$K_\alpha = \left[ \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} a_{nk} n^{\frac{2}{\alpha} - k} \prod_{l=0}^{k-1} (l - 2/\alpha) \right]^{-1} \quad (12)$$

and  $C_\alpha(\beta R^\alpha)^{\frac{2}{\alpha}} = \int_{\mathbb{R}^2} 1 - E[e^{-\zeta S_i |x|^{-\alpha}}] dx$ .

*Proof:* The proof is presented in Appendix A. ■

This Theorem has two key contributions and several implications. The Theorem's two contributions are (1) that it gives the exact probability of outage for any network density or target SIR (thermal noise was eliminated for convenience but could easily be reinserted, see Appendix A) but also (2) that it gives a solution for the optimal contention density in the low outage regime. As for consequences of the Theorem, first, it reinforces the linear dependence of the optimal contention density with the outage constraint for uniformly distributed random access systems. When compared with a regular network topology,  $\epsilon$  essentially becomes a penalty factor on the area spectral efficiency achievable with random access. Second, it shows that a large class of received signal and fading distributions is amenable to a transmission capacity analysis, including a number of MIMO techniques. Third, it demonstrates that derivation of the transmission capacity consists of two components: (1) determining  $K_\alpha$  which is dependent on the received signal distribution, and (2) determining  $C_\alpha$  which is a result of the interfering signal statistics. This holds in general only when the condition of independence between the received signal distribution and the interfering shot noise process is satisfied.

The results in Theorem 1 give fundamental limits on the operating point of a communicating pair and its performance in an interference-limited environment and there are several ways one could interpret these expressions. One interpretation is that for a communicating pair amidst a density  $\bar{\lambda}$  of interferers, the pair is free to choose any rate-outage-distance operating point for which  $\bar{\lambda} \leq \frac{K_\alpha \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2}$ . Furthermore, the operating point can be chosen independently of the operating

points of any other pair and hence the statement of the Theorem is very general.

On the other hand, if network-wide performance constraints  $\beta$  and  $\epsilon$  are imposed, then implicitly an upper limit on  $R$  is also established. If amidst a given density  $\bar{\lambda}$  a pair of nodes wish to communicate over a distance greater than  $R$ , then either outage probability or data rate must suffer. For clarity of the presentation, we will assume that  $\epsilon$ ,  $\beta$ , and  $R$  are fixed for every communicating pair. The analysis could be expanded by permitting a distribution on  $R$  in which case:

$$\bar{\lambda} = \frac{K_\alpha \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}}} \int \frac{1}{r^2} dF_R(r)$$

for which the maximum distance still permits the small outage approximation. It was shown in [16] that considering variable transmission distances has minimal impact on the transmission capacity. Specifically, the transmission capacity is reduced by the factor  $E[R^2]/E[R]^2$  when one imposes a distribution on  $R$ . Finally, the benefit in terms of transmission capacity to the network of the various MIMO techniques (embodied in the factors  $K_\alpha$  and  $C_\alpha$ ) remain unaffected by variable transmission distances.

### III. TRANSMISSION CAPACITY IN LOS AND NLOS ENVIRONMENTS

In [11], the same Poisson network model was used but propagation was modeled with path loss only while [24] incorporated Rayleigh fading in addition to path loss. In order to characterize the effect on network capacity between these extremes, Rayleigh fading and non-fading, let the envelope of the received signal be Nakagami- $m$  distributed with integer parameter  $m$  in addition to being scaled by path loss. The Nakagami distribution includes Rayleigh as a special case ( $m = 1$ ), non-fading as a special case ( $m = \infty$ ), and provides a close parameterized fit for empirical data as well as the Ricean distribution for  $m = \frac{(K+1)^2}{(2K+1)}$  for  $K$  the Ricean factor [26]. Theorem 1 is applied as follows:

*Proposition 1:* For a random access single-antenna narrowband wireless network in Nakagami- $m$  fading for  $m \in \mathbb{N}$ , the optimal contention density with outage  $\epsilon$  is given by

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha,m} \epsilon}{C_{\alpha,m} \beta^{\frac{2}{\alpha}} R^2} \quad (13)$$

where

$$K_{\alpha,m} = \left[ 1 + \sum_{k=0}^{m-2} \frac{1}{(k+1)!} \prod_{l=0}^k (l - 2/\alpha) \right]^{-1} \quad (14)$$

and

$$C_{\alpha,m} = \frac{2\pi}{\alpha} \sum_{k=0}^{m-1} \binom{m}{k} B\left(\frac{2}{\alpha} + k; m - \left(\frac{2}{\alpha} + k\right)\right) \quad (15)$$

with  $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$  being the Beta function. Further, both  $K_{\alpha,m}$  and  $C_{\alpha,m}$  increase as  $\Theta(m^{\frac{2}{\alpha}})$  with

$$m^{\frac{2}{\alpha}} \leq K_{\alpha,m} \leq \Gamma(1 - 2/\alpha) m^{\frac{2}{\alpha}} \quad (16)$$

and

$$0 < \frac{K_{\alpha,1}}{C_{\alpha,1}} < \frac{K_{\alpha,m}}{C_{\alpha,m}} < \frac{1}{\pi} \quad (17)$$

<sup>1</sup>Note that not all sets lead to valid distributions, e.g., for  $n = k = 1$ ,  $F^c(x) \propto xe^{-x}$  which cannot be a valid CCDF. Hence, the expressions given in the Theorem rely on a valid CCDF to be correct.

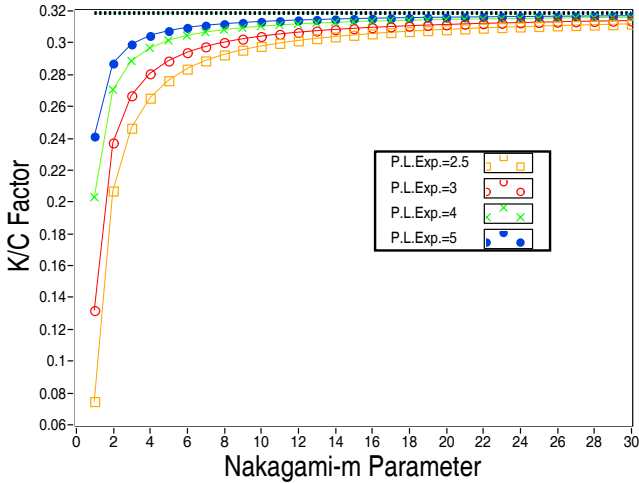


Fig. 1.  $\frac{K_{\alpha,m}}{C_{\alpha,m}} \rightarrow .318 \approx \frac{1}{\pi}$  as  $m \rightarrow \infty$  for various path loss exponents.

*Proof:* The proof is presented in Appendix B. ■

In [11] the transmission capacity in a non-fading environment is bounded above by  $\lambda_\epsilon \leq \frac{\epsilon}{\pi\beta^\frac{2}{\alpha}R^2}$  and further more this upper bound is fairly tight which implies that in fact  $\lim_{m \rightarrow \infty} \frac{K_{\alpha,m}}{C_{\alpha,m}} \approx \frac{1}{\pi}$ . Fig. 1 shows the ratio  $\frac{K_{\alpha,m}}{C_{\alpha,m}}$  for various  $\alpha$  versus  $m$ . This reinforces the (rough) tightness of the upper bound in [11].

Proposition 1 bridges the gap between fading and non-fading environments and demonstrates the potentially significant gain in network capacity relative to non-fading environments. It also shows that environments with lower path loss suffer more from severe fading (when in the common practical case  $\beta > 1$ ) and improve more with a strong LOS. The distinction is particularly important for dense networks communicating with nearby neighbors which are likely to have lower path loss *and* a significant LOS. The results also reveal the gains to be reaped by diversity techniques that can mitigate fading. The particular results for  $K_{\alpha,m}$  and  $C_{\alpha,m}$  will also be significant when analyzing MIMO techniques.

#### IV. SECTORIZED ANTENNAS

Now consider the same network model but with transmitters and receivers that are each equipped with  $M$  sectorized antennas. Let each antenna cover an angle of  $\frac{2\pi}{M}$  radians with an aperture gain of  $M$  for both transmitting and receiving in its sector and (potentially) with some small input/output gain outside its sector. Assume each transmitter picks a receiver in a uniformly random direction, and for each transmitter/receiver pair both know the sector in which to communicate with their intended partner. The model can include a constant sidelobe level  $\gamma$ , where the ratio of the sidelobe level to the main lobe is  $0 \leq \gamma \leq 1$ , for out-of-sector power which is both transmitted and received by the sectorized antenna. Fig. 2 depicts the model. The Table 1 conveys the power emitted by a transmitter in and out-of sector subject to constant total power  $\rho$ . Under this model, the following Proposition holds:

*Proposition 2: For a random access wireless network in which nodes have  $M$  sectorized directional antennas in Nakagami- $m$  fading with a constant (fractional) sidelobe level*

TABLE I  
RADIATED POWER DENSITIES

	In sector	Out of sector	Combined
Power emitted:	$\frac{\rho}{1+\gamma(M-1)}$	$\frac{\rho\gamma(M-1)}{1+\gamma(M-1)}$	Sum: $\rho$
Sector size (rad):	$2\pi\frac{1}{M}$	$2\pi\frac{M-1}{M}$	Sum: $2\pi$
Power density:	$\frac{M/2\pi}{1+\gamma(M-1)}$	$\frac{\gamma M/2\pi}{1+\gamma(M-1)}$	Ratio: $\gamma$

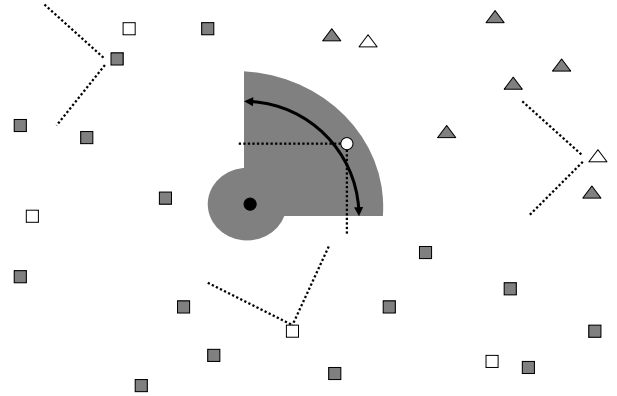


Fig. 2. Sectorized antenna model with a  $\frac{\pi}{2}$  sector main beam and constant sidelobe level for the receiver of interest (black dot), its intended transmitter (white dot), and the four sets of interferers:  $\Phi_1$  the interferers in the active sector transmitting toward the receiver (white triangles),  $\Phi_2$  the interferers in the active sector transmitting away from the receiver (shaded triangles),  $\Phi_3$  the interferers out of the active sector transmitting toward the receiver (white squares), and  $\Phi_4$  the interferers out of the active sector transmitting away from the receiver (shaded squares).

$\gamma \in [0, 1]$  for out of sector power transmitted and received, the optimal contention density with outage  $\epsilon$  is given by:

$$\bar{\lambda}_\epsilon = \left( \frac{M}{1 + \gamma^\frac{2}{\alpha}(M-1)} \right)^2 \frac{K_{\alpha,m}\epsilon}{C_{\alpha,m}\beta^\frac{2}{\alpha}R^2} \quad (18)$$

where  $K_{\alpha,m}$  is given by (14) and  $C_{\alpha,m}$  by (15). Thus  $\gamma^{-\frac{4}{\alpha}}$  is an upper bound on the transmission capacity increase due to antenna sectorization.

*Proof:* We assume the fading statistics of the received signal are unchanged by the sectorized antennas, but rather are merely scaled by the emitted and received power density. That the fading statistics are unchanged is reasonable for a small to moderate number of sectors, while for very directional antennas, the scattering seen by any given sector will be reduced and no longer have the typical isotropic properties. As a result of sectorization, four interference terms surface as follows: Let  $\Phi_1$  be the set of interferers which are in the active sector of the receiver of interest and are transmitting toward the receiver. Let  $\Phi_2$  be the set of interferers in the receiver's active sector which are not transmitting toward the receiver. Let  $\Phi_3$  be the set of interferers outside the receiver's sector which are transmitting toward the receiver. And let  $\Phi_4$  consist of those interferers transmitting away from the receiver and which are not in the receiver's sector. The independence property of the Poisson process implies these four shot noise processes are independent as well. Furthermore, the  $I_{\Phi_i}$  are each related to  $I_\Phi$  since they occur over disjoint subsets of the plane (i.e., a certain sector), are scaled by the combined

TABLE II  
INTERFERENCE SHOT NOISE PROCESSES

–	$\lambda_i$	$\theta_i$	$\psi_i$
$I_{\Phi_1}$	$\lambda \frac{1}{M}$	$2\pi \frac{1}{M}$	$\left(\frac{M}{1+\gamma(M-1)}\right)^2$
$I_{\Phi_2}$	$\lambda \frac{M-1}{M}$	$2\pi \frac{1}{M}$	$\gamma \left(\frac{M}{1+\gamma(M-1)}\right)^2$
$I_{\Phi_3}$	$\lambda \frac{1}{M}$	$2\pi \frac{M-1}{M}$	$\gamma \left(\frac{M}{1+\gamma(M-1)}\right)^2$
$I_{\Phi_4}$	$\lambda \frac{M-1}{M}$	$2\pi \frac{M-1}{M}$	$\gamma^2 \left(\frac{M}{1+\gamma(M-1)}\right)^2$

antenna gains, and the point process of interferers is thinned according to the direction the interferers transmit. The table summarizes the interference contributions from each of these processes with  $\lambda_i$  the effective node density of the process,  $\theta_i$  the sector size over which the process occurs from the perspective of the typical receiver, and  $\psi_i$  the combined antenna gains.

The transforms of the shot noise processes of the  $I_{\Phi_i}$  are given by

$$\mathcal{L}_{\Phi_i}(\zeta) = \exp \left\{ -\lambda_i \theta_i \int_0^\infty 1 - E \left[ e^{-\zeta \psi_i S |x|^{-\alpha}} \right] dx \right\} \quad (19)$$

for  $\zeta = \psi_0^{-1} \beta R^\alpha$  with  $\psi_0$  being the combined antenna gain between the typical receiver and its intended transmitter, and  $\psi_0 = \psi_1$ . Consider the Rayleigh fading case. The outage probability at a typical receiver is

$$\begin{aligned} \mathbf{P}(SIR \geq \beta) &= \mathbf{P} \left( \frac{\psi_0 \rho S_0 R^{-\alpha}}{\rho \sum_{i=1}^4 I_{\Phi_i}} \geq \beta \right) \\ &= \int_0^\infty F_{S_0}^c(\psi_0^{-1} \beta R^\alpha s) f_{[I_{\Phi_1} + I_{\Phi_2} + I_{\Phi_3} + I_{\Phi_4}]}(s) \\ &= \prod_{i=1}^4 \mathcal{L}_{\Phi_i}(\zeta) \Big|_{\zeta = \psi_0^{-1} \beta R^\alpha} \\ &= \exp \left\{ -\lambda \beta^{\frac{2}{\alpha}} R^2 C_\alpha \left( \frac{1 + \gamma^{\frac{2}{\alpha}} (M-1)}{M} \right)^2 \right\} \end{aligned} \quad (20)$$

and solving for  $\lambda$  gives

$$\bar{\lambda}_\epsilon = \left( \frac{M}{1 + \gamma^{\frac{2}{\alpha}} (M-1)} \right)^2 \frac{\epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2}. \quad (21)$$

Next note that  $\frac{M}{1 + \gamma^{\frac{2}{\alpha}} (M-1)} \leq \frac{1}{\gamma^{\frac{2}{\alpha}}}$  and as  $M$  becomes large, we have the limit

$$\lim_{M \rightarrow \infty} \left( \frac{M}{1 + \gamma^{\frac{2}{\alpha}} (M-1)} \right)^2 = \frac{1}{\gamma^{\frac{4}{\alpha}}}. \quad (22)$$

This results in an upper bound of  $\gamma^{-\frac{4}{\alpha}}$  on the improvement (over (13)) in optimal contention density from sectorized antennas. If signals are Nakagami- $m$  distributed instead, since the desired and interfering signals are independent,  $C_{\alpha, m}$  replaces  $C_\alpha$  and  $K_{\alpha, m}$  appears in the numerator of (21). ■

These results firstly indicate that directional antennas increase transmission capacity by nearly a factor of  $M^2$  for low sidelobe levels. This indicates that MIMO techniques that avoid or reduce interference in an ad hoc network are highly beneficial at the physical layer. In addition there are advantages at higher network layers such as increased ability

to learn the topology of the network, perform directional routing, etc; see [27] and [28] and references therein for more details. This section has characterized the potential increase in area spectral efficiency due to antenna sectorization which by itself provides greater potential and flexibility for routing and network management, but the full relationship between directional antennas and these higher layer functions is still an area of ongoing research.

However, this analysis also indicates that if for practical reasons, sidelobe levels cannot be reduced, then the sidelobes limit the potential gains even for very directional antennas. This model also suffers from very idealistic assumptions about the real propagation environment, especially since dense multipath can result in signal angle of arrival being quite different from the geographic angle to the transmitter. As pointed out in [27], real antenna patterns are far from “pie slices” and in multipath environments, static antennas are much less robust to fluctuating channels.

## V. TRANSMISSION CAPACITY OF EIGEN-BEAMFORMING NETWORKS

Dynamic beamforming is one of the most prominent multiple antenna techniques, having been employed for decades in electromagnetic detection and imaging applications. The complexity is manageable and it can be performed on any number of antennas in any configuration ([29], ch. 6). However, to be explicit since “beamforming” has become quite an overloaded term, this section uses the term to mean the following: At the receiver it refers to a coherent linear combination of the antenna outputs, while at the transmitter it refers to sending linearly weighted versions of the same signal on each antenna. Thus, unlike the previous section, no attention is paid to the specific physical pattern of energy propagation. In each case for this analysis, the weights are determined by the dominant singular vectors or eigenvectors (hence, “eigen-beamforming”) of the channel. Throughout this section it is assumed that both the transmitter and receiver have perfect channel knowledge of their own channel, but not of interfering channels. Hence, signaling strategies will maximize SNR over a specific channel but not necessarily SINR, though the analysis of the resulting interference-limited systems will ultimately ignore background thermal noise. We focus first on the vector (SIMO or MISO) channel for which eigen-beamforming is equivalent to maximal ratio transmission or combining. We then consider the general matrix (MIMO) channel for which a single datastream is sent over the dominant eigenmode.

### A. $1 \times M$ and $M \times 1$ Eigen-Beamforming

Consider first a wireless system in which all transmitters transmit with power  $\rho$  using only one antenna and receivers beamform on  $M$  antennas by coherently combining the received signals. Again, this is beamforming along the dominant (and only) eigenmode of the  $1 \times M$  channel. As shown in [18], this is equivalent to an  $M \times 1$  vector channel for which maximal ratio transmission is performed at the transmitter and one receive antenna is used. The channel model for the desired signal in a Rayleigh fading environment is a vector of i.i.d. unit variance, complex Gaussian entries scaled by

the power law path loss function:  $\mathbf{h}_0 \sqrt{|R|^{-\alpha}}$  for the  $k$ th entry of  $\mathbf{h}_0$  independently  $[\mathbf{h}_0]_k \sim \mathcal{CN}(0, 1)$ , and similarly the channel between a receiver and the  $i$ th interferer is  $\mathbf{h}_i \sqrt{|X_i|^{-\alpha}}$  with  $[\mathbf{h}_i]_k \sim \mathcal{CN}(0, 1)$ . Under this model, the following Proposition holds. As in Sec. III, the Proposition will be given in two parts: the first is an expression for the exact optimal contention density for small outage constraints and the second is a set of bounds that help interpret the exact results.

*Proposition 3: For a random access wireless network in which nodes transmit on a single antenna and perform maximal ratio combining with  $M$  antennas; or equivalently perform maximal ratio transmission with  $M$  antennas and receive on a single antenna; the optimal contention density under Rayleigh fading with outage constraint  $\epsilon$  is given by:*

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha, M} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \quad (23)$$

where  $K_{\alpha, M}$  is given by (14) and  $C_\alpha = C_{\alpha, 1}$  in (15). Further,  $\bar{\lambda}_\epsilon$  is  $\Theta(M^{\frac{2}{\alpha}})$  and bounded by:

$$\frac{M^{\frac{2}{\alpha}} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \leq \bar{\lambda}_\epsilon \leq \frac{\Gamma(1 - \frac{2}{\alpha}) M^{\frac{2}{\alpha}} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2}. \quad (24)$$

*Proof:* To characterize the interference seen by an  $M$ -antenna receiver that ignores interfering signals, beamforming simply to maximize its own received signal power (again thermal noise is assumed negligible), the SIR expression is:

$$\begin{aligned} SIR &= \frac{\frac{\rho}{M} |\mathbf{h}_0^H \mathbf{h}_0|^2 R^{-\alpha}}{\frac{\rho}{M} \sum_{X_i \in \Phi} |\mathbf{h}_0^H \mathbf{h}_i|^2 |X_i|^{-\alpha}} \\ &= \frac{\|\mathbf{h}_0\|^2 R^{-\alpha}}{\sum_{X_i \in \Phi} \left| \frac{\mathbf{h}_0^H \mathbf{h}_i}{\|\mathbf{h}_0\|} \right|^2 |X_i|^{-\alpha}}. \end{aligned} \quad (25)$$

As shown in [17], since a linear combination of Gaussian variables is again Gaussian, the product  $\frac{\mathbf{h}_0^H \mathbf{h}_i}{\|\mathbf{h}_0\|}$  is distributed as a single complex Gaussian random variable with zero mean and unit variance. Letting  $S_i = \left| \frac{\mathbf{h}_0^H \mathbf{h}_i}{\|\mathbf{h}_0\|} \right|^2$ , which is exponentially distributed, the SIR expression is

$$SIR = \frac{\|\mathbf{h}_0\|^2 R^{-\alpha}}{\sum_{X_i \in \Phi} S_i |X_i|^{-\alpha}} = \frac{\|\mathbf{h}_0\|^2 R^{-\alpha}}{I_\Phi}. \quad (26)$$

Setting  $S_0 = \|\mathbf{h}_0\|^2$  and considering the network model in Sec. II but with beamforming receivers with  $M$  antennas, the distribution of the received signal is now  $\chi^2$  with  $2M$  degrees of freedom. The CCDF of  $S_0$  is  $F_{S_0}^c(x) = e^{-x} \sum_{k=0}^{M-1} \frac{x^k}{k!}$ . However, the interference has the same form as the shot noise process for the single-antenna case. If we now apply a small outage constraint and Theorem 1, we can state simply that  $K_{\alpha, M}$  is given by (14) and  $C_\alpha = C_{\alpha, 1}$  in (15). As shown in (16),

$$1 \leq \frac{K_{\alpha, M}}{M^{\frac{2}{\alpha}}} \leq \Gamma(1 - 2/\alpha)$$

which indicates that

$$\frac{M^{\frac{2}{\alpha}} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \leq \frac{K_{\alpha, M} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \leq \frac{\Gamma(1 - \frac{2}{\alpha}) M^{\frac{2}{\alpha}} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \quad (27)$$

with equality to the lower bound at  $M = 1$  and approaching the upper bound with increasing  $M$  since  $C_\alpha$  is constant while

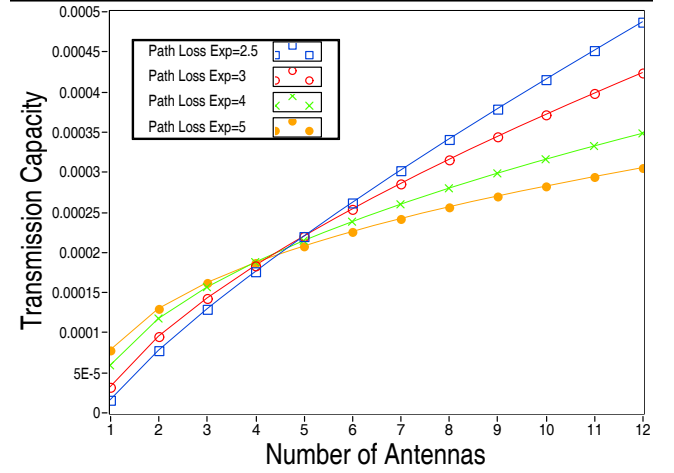


Fig. 3. Transmission capacity versus  $M$  for four path loss exponents for  $1 \times M$  MRC. Higher path loss separates transmissions spatially and is the dominant effect for smaller numbers of antennas. But with a larger number of antennas, ultimately network performance is improved more through interference robustness than spatial separation.

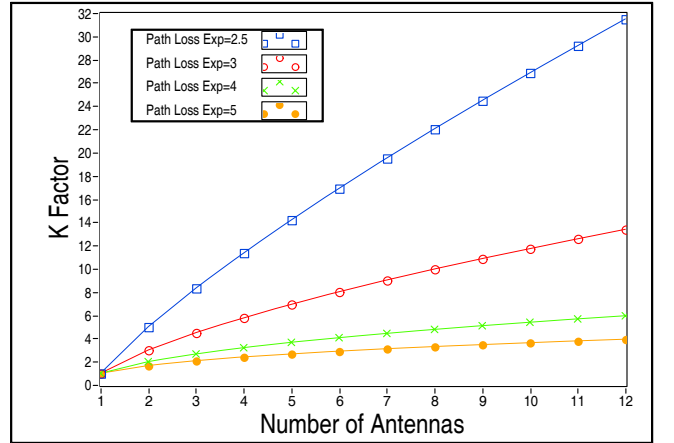


Fig. 4. The  $K_{\alpha, M}$  factor versus  $M$  for four path loss exponents for  $1 \times M$  MRC. Lower path loss results in much greater gains over the SISO case ( $M = 1$ ).

$K_{\alpha, M}$  is increasing in  $M$ . The term in the middle is now equal to  $\bar{\lambda}_\epsilon$ . ■

Proposition 3 gives a general scaling of the optimal contention density with the number of antennas, target SIR, path loss, the transmitter-receiver separation, and the outage constraint. Fig. 3 gives the transmission capacity versus  $M$  for four different path loss exponents. Fig. 4 gives the  $K_{\alpha, M}$  factor versus  $M$  for the same path loss exponents. As evident from the figures, as path loss reduces and interference becomes less attenuated by distance, the gain of the MIMO technique over the SISO case increases. However, higher path loss results in higher transmission capacity for smaller numbers of antennas since path loss helps to spatially separate transmissions. Fig. 5 demonstrates the relationship of the exact  $K_{\alpha, M}$  factor to the upper and lower bounds. The upper bound is both asymptotically tight and a good approximation for higher path loss.

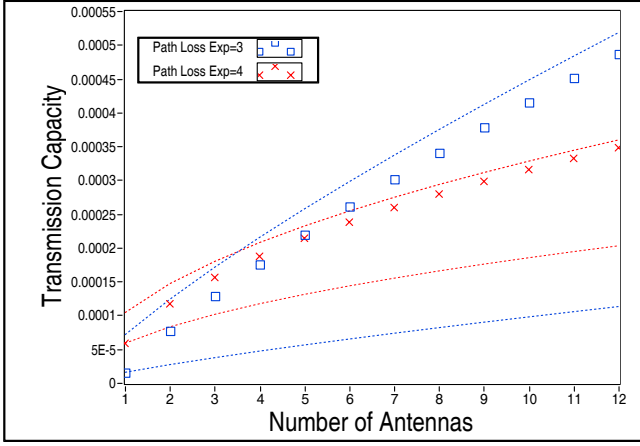


Fig. 5. Demonstration of the bounds on transmission capacity for  $1 \times M$  or equivalently  $M \times 1$  MRC. The upper bound is asymptotically tight and a good approximation for higher path loss.

### B. $M_t \times M_r$ MIMO Eigen-Beamforming

Now consider the same network but with nodes each equipped with  $M_t$  transmit and  $M_r$  receive antennas to perform dynamic eigen-beamforming at both transmitter and receiver ends. This extension of MRC has significant advantages even over  $1 \times M$  MRC since the diversity order increases as  $M_t M_r$ . The Rayleigh fading MIMO channel is modeled as a matrix of i.i.d. zero-mean, unit-variance complex Gaussian entries scaled by path loss. The channel of the desired signal for the transmitter-receiver pair of interest is denoted  $\mathbf{H}_{00}$ . The transmitter and receiver beamform using the input and output singular vectors  $\mathbf{v}_0$  and  $\mathbf{u}_0$ , respectively, corresponding to the maximum singular value of  $\mathbf{H}_{00}$ . This results in the received power being equal to the square of the maximum singular value  $\phi_{\max}^2$  scaled by path loss and the transmit power. Each interfering transmitter, on the other hand, beamforms to maximize received power across some other Rayleigh channel  $\mathbf{H}_{ii}$  using beamforming vector  $\mathbf{v}_i$ , and interferes at the receiver of interest through channel  $\mathbf{H}_{0i}$ . For such a network, the following bounds hold:

*Proposition 4:* For a random access wireless network in which nodes perform maximal ratio transmission and combining on  $M_t$  and  $M_r$  antennas respectively, for small outages the optimal contention density is bounded by:

$$\frac{\max\{M_t, M_r\}^{\frac{2}{\alpha}} \epsilon}{C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}}} \leq \bar{\lambda}_\epsilon \leq \frac{\Gamma(1 - \frac{2}{\alpha})(M_t M_r)^{\frac{2}{\alpha}} \epsilon}{C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}}} \quad (28)$$

for  $C_\alpha = C_{\alpha,1}$  given in (15).

*Proof:* To begin, the SIR expression for this model is

$$SIR = \frac{\rho \phi_{\max}^2 R^{-\alpha}}{\rho \sum_{X_i \in \Phi} |X_i|^{-\alpha} |\mathbf{u}_0^H \mathbf{H}_{0i} \mathbf{v}_i|^2}. \quad (29)$$

Note that  $\mathbf{u}_0$ ,  $\mathbf{H}_{0i}$ , and  $\mathbf{v}_i$  are all independent. As discussed in [19], the full product  $\mathbf{u}_0^H \mathbf{H}_{0i} \mathbf{v}_i$  is distributed as a single zero-mean, unit-variance, complex Gaussian variable since the inner product of a vector i.i.d. complex Gaussian variables with an arbitrary unit vector is a single complex Gaussian variable. This simplifies the SIR expression to  $SIR = \frac{\phi_{\max}^2 R^{-\alpha}}{I_\Phi}$ , with the distribution of the interference unchanged from the single antenna Rayleigh fading case. Again neglecting thermal

noise, the received and interfering signals are independent and  $C_\alpha = C_{\alpha,1}$  in (15) by equivalence of the shot noise processes.

As for the received signal, note that the CCDF of the square of the maximum singular value of the desired channel (or equivalently the largest eigenvalue of a complex Wishart matrix), has been reported by Kang and Alouini [18] (originally given by Khatri [30]):

$$F_{\phi_{\max}^2}^c(x) = 1 - \frac{|\Psi(x)|}{\prod_{k=1}^q \Gamma(q-k+1) \Gamma(s-k+1)}$$

where  $|\cdot|$  denotes a determinant,  $q = \min\{M_t, M_r\}$ ,  $s = \max\{M_t, M_r\}$ , and the entries of the  $q \times q$  matrix  $\Psi(x)$  are given by  $\{\Psi(x)\}_{i,j} = \gamma(s-q+i+j-1, x)$ ,  $i, j = 1, \dots, q$  where  $\gamma(\cdot, \cdot)$  is the lower incomplete gamma function. Recall  $\gamma(n, x) = (n-1)! \left(1 - e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}\right)$ , for  $n \in \mathbb{N}$ . This now facilitates the application of Theorem 1 yielding the outage probability and the optimal contention density, which will again have the form:

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha, M_t, M_r}^{mrt} \epsilon}{C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}}}.$$

We are unable to give an expression for  $K_{\alpha, M_t, M_r}^{mrt}$  since the explicit sum-of-exponentials-and-polynomials form for  $F_{\phi_{\max}^2}^c$  is not known. However, the largest squared singular value is bounded by [31]:

$$\|\mathbf{H}_{00}\|_F^2 \geq \phi_{\max}^2 \geq \frac{\|\mathbf{H}_{00}\|_F^2}{\min\{M_t, M_r\}}.$$

Since  $\|\mathbf{H}_{00}\|_F^2$  is  $\chi^2$  with  $2M_t M_r$  degrees of freedom this is equivalent to a particular MRC case in (23) and (24) indicating that

$$K_{\alpha, M_t, M_r}^{mrt} \leq K_{\alpha, (M_t M_r)} \leq \Gamma(1 - 2/\alpha) (M_t M_r)^{\frac{2}{\alpha}} \quad (30)$$

so that

$$\bar{\lambda}_\epsilon \leq \frac{\Gamma(1 - \frac{2}{\alpha})(M_t M_r)^{\frac{2}{\alpha}} \epsilon}{C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}}}. \quad (31)$$

Furthermore, a lower bound can be obtained from (24) as

$$\begin{aligned} K_{\alpha, M_t, M_r}^{mrt} &\geq \frac{K_{\alpha, (M_t M_r)}}{\min\{M_t, M_r\}^{\frac{2}{\alpha}}} \\ &\geq \frac{(M_t M_r)^{\frac{2}{\alpha}}}{\min\{M_t, M_r\}^{\frac{2}{\alpha}}} \\ &= \max\{M_t, M_r\}^{\frac{2}{\alpha}} \end{aligned} \quad (32)$$

so that

$$\bar{\lambda}_\epsilon \geq \frac{\max\{M_t, M_r\}^{\frac{2}{\alpha}} \epsilon}{C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}}}. \quad (33)$$

Since  $K_{\alpha, M_t, M_r}^{mrt}$  cannot be given explicitly for arbitrary  $M_t$  and  $M_r$  at present, consider as an example the case  $M_t = M_r = 2$  for which

$$F_{\phi_{\max}^2}^c(x) = 2e^{-x} - e^{-2x} + x^2 e^{-x}.$$

Applying Theorem 1 for small outages:

$$\mathbf{P}(SIR < \beta) \approx \lambda C_\alpha R^2 \beta_\alpha^{\frac{2}{\alpha}} \left(2 - 2^{\frac{2}{\alpha}} - \frac{2}{\alpha} \left(1 - \frac{2}{\alpha}\right)\right). \quad (34)$$

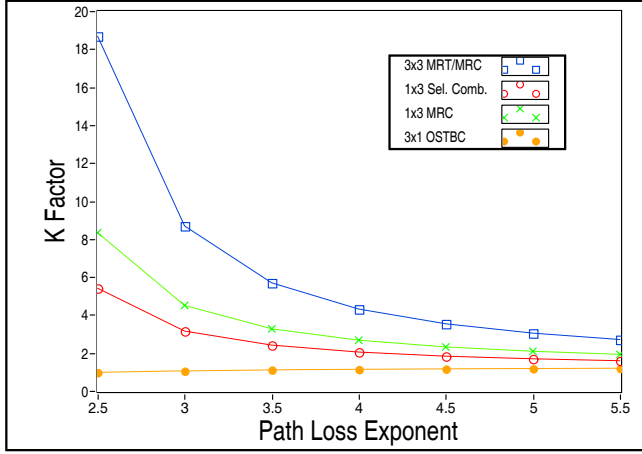


Fig. 6.  $K_{\alpha,3}^{mrt}$  ( $3 \times 3$  MRT/MRC),  $K_{\alpha,3}$  ( $1 \times 3$  MRC),  $K_{\alpha,M}^{sc}$  ( $1 \times 3$  selection combining), and  $\frac{K_{\alpha,M_t}}{C_{\alpha,M_t}} C_{\alpha,1}$  ( $3 \times 1$  OSTBC) versus  $\alpha$  for various  $M$ . The factor for OSTBCs is the only which increases (slightly) with increasing path loss since interference reduces. (The factor  $C_{\alpha,1}$  is a normalizing factor for fair comparison with other techniques and the SISO case for which  $K = 1$ .)

For  $M_t = M_r$  and both large,  $\phi_{\max}^2$  of the channel matrix approaches  $4M_t$  [32]. This leads to the conjecture that for moderately large numbers of antennas (e.g.,  $M_t, M_r > 3$ ), the lower bound reflects the orderwise behavior in a rich scattering environment. However, the upper bound should be more appropriate in a LOS channel. Fig. 6 depicts both  $K_{\alpha,M}$  and for the square channel case  $M = M_t = M_r$ ,  $K_{\alpha,M}^{mrt}$  for various  $M$ ,  $\alpha$ . Again  $K_{\alpha,M}^{mrt} \rightarrow 1$  with increasing  $\alpha$ . This implies that as  $\alpha$  becomes large, nodes are already spatially separated through path loss, and spatial diversity yields less improvement over the single antenna case.

We also point out that the expression for  $F_{\phi_{\max}^2}^c$  in Rayleigh fading channels, repeated above from [18], is given for any number of antennas at either the transmitter or receiver. The result is always a sum of terms of the form  $x^k e^{-nx}$  so that the Laplace transform method used here may be applied for systems with any number of antennas. For larger numbers of antennas especially, there also remains the question if spatial multiplexing has a place. While this is left for future investigation, it should be noted that the form of the joint distribution of the eigenvalues of Wishart matrices given in [33] indicates that the Laplace transform method can be extended to the spatial multiplexing case.

## VI. TRANSMISSION CAPACITY OF OSTBC NETWORKS

Orthogonal space-time block coding has been one of the more quickly accepted transmit diversity techniques for several good reasons. First, OSTBCs achieve full diversity in point-to-point links without requiring channel state information at the transmitter. Second, an optimum receiver design is simply a matched filter without any need for joint decoding of multiple symbols (error correction codes notwithstanding). Furthermore, space-time coding results in far less variability in the effective channel, greatly reducing the frequency and duration of deep fades. However, there is another source of effective channel instability, particularly in decentralized networks, which is cochannel interference. In light of the

results on reduced fading as well as MRT/MRC earlier in this paper, for which the latter results in much greater network improvement, it is unclear how OSTBCs compare in a decentralized, interference-limited environment and warrants further investigation.

Specific codes are characterized by the number of transmit antennas used ( $M_t$ ), the number of time slots used ( $N$ ), and the number of independent data symbols sent ( $N_s$ ) [34]. Again  $M_r$  denotes the number of receive antennas but this has no effect on the code structure. However, it will also be necessary to characterize OSTBCs by the number of time slots over which each symbol is repeated ( $N_r$ ). The familiar Alamouti code has  $M_t = N = N_s = N_r = 2$ .

*Proposition 5: For a random access wireless network in which transmitting nodes use orthogonal space-time block codes with  $M_t$  transmit antennas and code parameter  $N_r$  and receiving nodes perform maximal ratio combining with  $M_r$  antennas in Rayleigh fading, the optimal contention density under the outage constraint  $\epsilon$  is given by:*

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha, M_t M_r} \epsilon}{C_{\alpha, N_r} \beta^{\frac{2}{\alpha}} R^2} \quad (35)$$

where  $K_{\alpha, M_t M_r}$  is given by (14) and  $C_{\alpha, N_r}$  in (15). Further,  $\bar{\lambda}_\epsilon$  is  $\Theta(M_r^{\frac{2}{\alpha}})$ :

$$\frac{M_r^{\frac{2}{\alpha}} \epsilon}{C_{\alpha, 1} \beta^{\frac{2}{\alpha}} R^2} \leq \bar{\lambda}_\epsilon \leq \frac{M_r^{\frac{2}{\alpha}} \epsilon}{\pi \beta^{\frac{2}{\alpha}} R^2}. \quad (36)$$

*Proof:* For the received signal, since detection decouples for OSTBCs [35] over the  $N_r$  time slots as well as  $M_r$  antennas, the received amplitude is  $\|\mathbf{H}_0\|_F^2$  for each symbol [35], where  $\mathbf{H}_0$  is the  $M_r \times M_t$  complex Gaussian channel. The distribution of  $\|\mathbf{H}_0\|_F^2$  is  $\chi^2$ , just as with MRC, but with  $2M_t M_r$  degrees of freedom. So applying Theorem 1, the  $K$  factor is  $K_{\alpha, M}$  in (14) for  $M = M_t M_r$ .

The interference seen by an OSTBC processing system is more complicated, however. To determine the distribution of  $S_i$ , consider the expression for the interference term from a single interferer:

$$|X_i|^{-\alpha} S_i = |X_i|^{-\alpha} \sum_{k=1}^{N_r} \frac{\mathbf{h}_0^H}{\|\mathbf{h}_0\|} \mathbf{h}_i^{(k)} s_i^{(k)} = |X_i|^{-\alpha} \sum_{k=1}^{N_r} S_i^{(k)} \quad (37)$$

where  $\mathbf{h}_0 = \text{vec}(\mathbf{H}_0)$  and  $\mathbf{h}_i^{(k)}$  is a permutation of the entries in  $\text{vec}(\mathbf{H}_i)$  depending on the block coding structure. Since desired symbols are repeated  $N_r$  times, each  $S_i$  is a sum  $N_r$  terms  $S_i^{(k)}$  each of which is exponentially distributed though not independent. In a strict sense, this violates the independence of  $S_0$  and  $S_i$  required by Theorem 1. However, we assume rough independence of received and interfering signal statistics, with the statistics of the sum of  $S_i^{(k)}$  nearly indistinguishable from a Gamma distribution independent of  $S_0$ . The nature of the post-processing interference here was also reported in [36]. Note that this assumption removes some inherent structure in the interference so that the analysis becomes worst case.

Since the Gamma distribution is the same mark distribution encountered for Nakagami- $m$  fading interferers,  $C_{\alpha, N_r}$  is given by (15). As shown before, the factor increases with

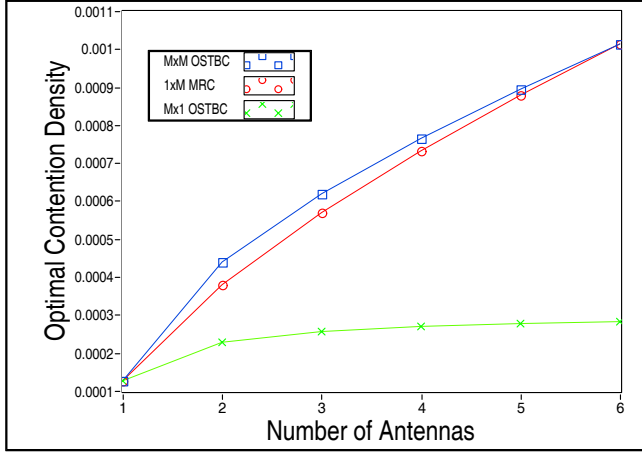


Fig. 7. Comparison of the optimal contention density for  $M \times M$  OSTBC/MRC,  $1 \times M$  MRC, and  $M \times 1$  OSTBC respectively. The optimal contention density is nearly identical to receiver MRC only and does not improve much for large numbers of transmit antennas over the single antenna case without receive beamforming.

$N_r^{\frac{2}{\alpha}}$  indicating that repeating the symbols introduces more cochannel interference. Applying Theorem 1 for small outages

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha, M_t M_r} \epsilon}{C_{\alpha, N_r} R^2 \beta^{\frac{2}{\alpha}}} \leq \frac{(M_t M_r)^{\frac{2}{\alpha}} \epsilon}{\pi N_r^{\frac{2}{\alpha}} R^2 \beta^{\frac{2}{\alpha}}} = \frac{M_r^{\frac{2}{\alpha}} \epsilon}{\pi R^2 \beta^{\frac{2}{\alpha}}} \quad (38)$$

for most practical block codes since  $M_t = N_r$ , which is the best case. For the lower bound, we simply ignore the change in the constant  $C_{\alpha, N_r}$  substituting  $C_{\alpha, 1}$  which is greater than  $\pi$ . (That is, let  $K_{\alpha, M}$  increase but not  $C_{\alpha, N_r}$ ). ■

The primary insight from the analysis of OSTBCs is that in an environment of significant cochannel interference, they accomplish little. As is evident from the bounds, the number of receive antennas is the primary factor in network performance. While block codes harden the channel resulting in a network performance gain equivalent to that gained from reducing fading, they also tend to amplify interference since symbols are repeated. When symbols are repeated multiple times from the same antenna, as in some orthogonal designs, this effect is worsened so that  $N_r = M_t$  is the best case. Furthermore, even though power is split between simultaneously transmitted symbols, the transmit antennas become multiple independent interference sources for other nodes in the network. Furthermore, OSTBCs take a hit in the data rate for any code beside Alamouti's. So for a larger number of antennas, OSTBCs are typically inferior to other schemes and are likely not worth even the slight added complexity.

Fig. 7 compares the optimal contention density for  $M \times 1$  OSTBCs for which the receiver receives on only one antenna,  $M \times M$  OSTBCs for which the receiver performs MRC on  $M$  antennas in addition to the transmit block coding, as well as  $1 \times M$  MRC without block coding. The figure shows the optimal contention density for  $\alpha = 3$ , target SINR 4.77dB, and transmitter-receiver separation 10m. First, there is little gain over simply performing MRC in contention density. But what is not shown is that for the number of antennas larger than two, the transmission capacity for  $M \times M$  OSTBCs actually falls below the MRC curve since it must use a reduced rate code. If only one receive antenna is used, then for any number

of transmit antennas beyond two, there is essentially no gain when code rate is taken into account. This confirms that the primary source of gain is at the receiver and that for any system beyond  $2 \times 2$ , it would be better to simply select one antenna and operate in the  $1 \times M$  MRC mode.

## VII. TRANSMISSION CAPACITY OF SELECTION DIVERSITY AND COMBINING NETWORKS

A fundamental characteristic of MIMO fading channels is that due to polarization, pattern diversity, or spatial separation, one or more antenna elements may be receiving above average signal strength. Simply selecting the best often has the practical advantage over more sophisticated combining schemes of simpler implementation, or less expensive hardware. There are a variety of ways to perform antenna selection, and antenna selection can be used in conjunction with other diversity techniques. As an example let the transmitter operate one antenna and the receiver select one of  $M$  which has the best instantaneous channel with i.i.d. Rayleigh fading between all antennas. In this case,

$$F_{S_0}^c(x) = 1 - (1 - e^{-x})^M = \sum_{k=1}^M \binom{M}{k} (-1)^{k+1} e^{-kx} \quad (39)$$

and the interference fading channels  $S_i$  remain exponentially distributed. This can be extended by considering a system that selects the best pair of antennas (one transmit and one receive) from among  $M_t$  transmit and  $M_r$  receive antennas. The parameter  $M$  in (39) is simply replaced by  $M_t M_r$ . Here the full matrix channel is  $\mathbf{H}_{00}$  as in Sec. V-B from which the element with the largest magnitude is selected. The following Proposition characterizes the gain from selection diversity:

*Proposition 6: For a random access wireless network in which nodes perform selection diversity/combining by selecting the best pair among  $M_t$  transmit and  $M_r$  receive antennas in Rayleigh fading, the optimal contention density under the outage constraint  $\epsilon$  is given by:*

$$\bar{\lambda}_\epsilon = \frac{K_{\alpha, M^2}^{sc} \epsilon}{C_\alpha \beta^{\frac{2}{\alpha}} R^2} \quad (40)$$

for  $M^2 = M_t M_r$ ,  $C_\alpha = C_{\alpha, 1}$  in (15), and

$$K_{\alpha, M^2}^{sc} = \left[ \sum_{k=1}^{M^2} \binom{M^2}{k} (-1)^{k+1} k^{\frac{2}{\alpha}} \right]^{-1}. \quad (41)$$

*Proof:* This is given by simply substituting the coefficients in (39) into Theorem 1 and noting that the statistics of the interference are identical to the SISO case for any pair of antennas. ■

There are a number of other distributions resulting from antenna selection that can be considered. For example, in an  $M_r \times M_t$  system performing MRC at the receiver, one transmit antenna may be selected which has the largest magnitude vector channel to the intended receiver. The distribution of the interference after MRC processing will remain the same

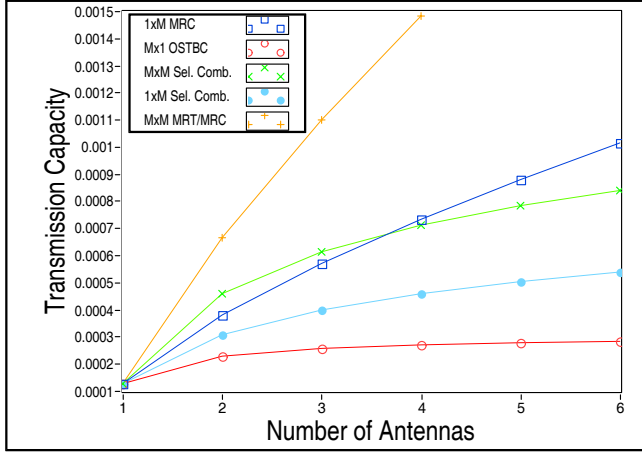


Fig. 8. Comparison of the transmission capacity for MRT, MRC, OSTBC, and selection combining. Note that  $M \times M$  OSTBC/MRC is nearly identical to the  $1 \times M$  MRC case. Also note that  $M \times M$  selection combining means selecting the best pair of transmitting and receiving antennas (one of each).

but  $S_0$  will have

$$\begin{aligned} F_{S_0}^c(x) &= 1 - \left( 1 - e^{-x} \sum_{k=0}^{M_r-1} \frac{x^k}{k!} \right)^{M_t} \\ &= \sum_{m=1}^{M_t} e^{-mx} \sum_{k=0}^{M_t(M_r-1)} a_{mk} x^k \end{aligned} \quad (42)$$

for

$$a_{mk} = (-1)^{M_t+m} \binom{M_t}{m} \sum_{\substack{n_1, n_2, \dots, n_m \leq M_r-1 \\ n_1+n_2+\dots+n_m=k}} \prod_{i=1}^m (n_i!)^{-1} \quad (43)$$

with the sum running over all (ordered)  $m$ -tuples of positive integers less than  $M_r - 1$  which add to  $k$ . From Theorem 1 the  $K$  factor can now be determined which specifies the optimal contention density as well. Fig. 8 compares the gain in transmission capacity for a number of systems versus the number of antennas, including MRT/MRC, OSTBCs, as well as two kinds of selection diversity/combining: one in which the transmitter transmits on one antenna and the receiver selects the best of its own antennas, and the second in which the receiver and transmitter jointly select the best pair of single antennas. Clearly antenna selection can significantly enhance network performance since it improves the typical channel without amplifying interference. Of course, as the number of antennas becomes large, obtaining  $M^2$  statistically independent pairs is difficult, and array gain quickly becomes superior in terms of network performance. Still, Proposition 6 implies that antenna selection may be a desirable tradeoff in terms of performance and complexity.

## VIII. CONCLUSION

In this paper, the performance of random access ad hoc networks employing a number of spatial diversity techniques was determined. Exact outage probabilities were derived for random wireless networks as well as optimal contention densities for small outage constraints for a large class of received signal distributions. These distributions include those

applicable for nodes employing maximal ratio transmission/combining, orthogonal space-time block coding, selection diversity/combining, and static beamforming with sectorized antennas. The improvement in transmission capacity for Nakagami- $m$  fading channels in which fading is reduced were also given and shown to be equivalent to the small gains due to space-time codes. The results show a significant improvement in transmission capacity for sectorized antennas and beamforming systems, a lesser but still appreciable gain for selection combining systems, and marginal gains at best for space-time block coded systems. Gains are higher for beamforming and selection combining systems when interference is more severe both when node density increases and under lower path loss, while the opposite is true for space-time block coded systems. In general it was found that diversity techniques employed at the receiver offer the most practical benefits. Future research should address the enhancements achievable from spatial multiplexing, multiuser MIMO techniques, and the combined effect of MIMO at the physical layer for scheduling, routing, and network management applied to ad hoc networks.

## APPENDIX A PROOF OF THEOREM 1

Define the PDF of  $I_\Phi$  to be  $f_{I_\Phi}(t) = d\mathbf{P}(I_\Phi \leq t)$ . Define a transform of  $f_{I_\Phi}(t)$  using CCDF  $F_S^c(t)$  as

$$\mathcal{G}_{I,S}(s) = \int_0^\infty F_S^c(st) f_{I_\Phi}(t) dt.$$

The probability of successful transmission can be expressed as

$$\begin{aligned} \mathbf{P}(SIR \geq \beta) &= \mathbf{P}(S \geq \beta R^\alpha I_\Phi) \\ &= \int_0^\infty F_S^c(st) f_{I_\Phi}(t) dt \\ &= \mathcal{G}_{I,S}(s) |_{s=\beta R^\alpha} \end{aligned} \quad (44)$$

When  $S \sim \text{Exp}(1)$ ,  $F_S^c(t) = e^{-t}$  so that the transform of  $f_{I_\Phi}(t)$  is

$$\begin{aligned} \mathcal{G}_{I,S}(s) &= \int_0^\infty F_S^c(st) f_{I_\Phi}(t) dt \\ &= \mathcal{L}\{f_{I_\Phi}(t)\}(s) = \mathcal{L}_{I_\Phi}(s), \end{aligned} \quad (45)$$

and the transmission success probability is expressible in terms of the Laplace transform. Next suppose  $F_S^c(t) = \sum_n e^{-nt} \sum_k a_{nk} t^k$ , then the transform of  $f_{I_\Phi}(t)$  using CCDF  $F_S^c(t)$  is

$$\begin{aligned} \mathcal{G}_{I,S}(s) &= \int_0^\infty F_S^c(st) f_{I_\Phi}(t) dt \\ &= \int_0^\infty \left( \sum_n e^{-nst} \sum_k a_{nk} (st)^k \right) f_{I_\Phi}(t) dt \end{aligned} \quad (46)$$

$$= \sum_n \sum_k a_{nk} s^k \left( \int_0^\infty e^{-nt} t^k f_{I_\Phi}(t) dt \right) \quad (47)$$

$$= \sum_n \sum_k a_{nk} s^k \mathcal{L}\{t^k f_{I_\Phi}(t)\}(ns) \quad (48)$$

$$= \sum_n \sum_k a_{nk} (-s)^k \frac{d^k}{d(ns)^k} \mathcal{L}_{I_\Phi}(ns) \quad (49)$$

$$= \sum_n \sum_k \left[ a_{nk} \left( -\frac{\zeta}{n} \right)^k \frac{d^k}{d\zeta^k} \mathcal{L}_{I_\Phi}(\zeta) \right]_{\zeta=n\beta R^\alpha} \quad (50)$$

where (48) uses the Laplace transform property  $t^n f(t) \longleftrightarrow (-1)^n \frac{d^n}{ds^n} \mathcal{L}[f(t)](s)$ . To derive (12), the derivatives of  $\mathcal{L}_{I_\Phi}$  are needed and they are given by

$$\frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi}(\zeta) = \frac{e^{-\lambda \zeta^{\frac{2}{\alpha}} C_\alpha}}{(-\zeta)^p} \sum_{k=1}^p \left( \lambda \zeta^{\frac{2}{\alpha}} C_\alpha \frac{2}{\alpha} \right)^k (-1)^k \Upsilon_{p,k} \quad (51)$$

where

$$\Upsilon_{p,k} = \sum_{\delta_j \in \text{comb}(\binom{p-1}{p-k})} \prod_{l_{ij} \in \delta_j} \left( \frac{2}{\alpha} (l_{ij} - i + 1) - l_{ij} \right),$$

$$i = 1, 2, \dots, |\delta_j|, j = 1, 2, \dots, \binom{p-1}{p-k}.$$

Here we define  $\text{comb}(\binom{a}{b})$  as the set of all subsets of the natural numbers  $\{1, 2, \dots, a\}$  of cardinality  $b$  with distinct elements, i.e.,  $\text{comb}(\binom{a}{b})$  is the set of combinations of  $\{1, 2, \dots, a\}$  taken  $b$  at a time. Thus there are  $\binom{a}{b}$  subsets in  $\text{comb}(\binom{a}{b})$  each with  $b$  elements and the  $\delta_j$  each constitute one such subset.

Forming the first order Taylor expansion for the  $p$ th derivative around  $\kappa = \lambda \beta^{\frac{2}{\alpha}} R^2 C_\alpha = 0$ , note that any term with  $\kappa^k$  for  $k > 1$  is  $o(\kappa)$  and can be discarded so that  $\left. \frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi}(\zeta) \right|_{\zeta=\beta R^\alpha}$  reduces to

$$\begin{aligned} \frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi}(\zeta) &= (-1)^{p+1} \frac{e^{-\lambda \zeta^{\frac{2}{\alpha}} C_\alpha}}{\zeta^p} \lambda \zeta^{\frac{2}{\alpha}} C_\alpha (2/\alpha) \Upsilon_{p,1} \\ &\quad + \Theta(\kappa^2) \\ &= (-1)^p e^{-\lambda \zeta^{\frac{2}{\alpha}} C_\alpha} \lambda \zeta^{\frac{2}{\alpha}-p} C_\alpha \prod_{l=0}^{p-1} (l - 2/\alpha) \\ &\quad + \Theta(\kappa^2) \\ &= (-1)^p \lambda \zeta^{\frac{2}{\alpha}-p} C_\alpha \prod_{l=0}^{p-1} (l - 2/\alpha) + \Theta(\kappa^2) \end{aligned} \quad (52)$$

where the small error terms are the result of the Taylor expansion. Thus a term from (10) becomes:

$$a_{nk} \left( -\frac{\zeta}{n} \right)^k \frac{d^k}{d\zeta^k} \mathcal{L}_{I_\Phi}(\zeta) = a_{nk} n^{\frac{2}{\alpha}-k} \lambda \zeta^{\frac{2}{\alpha}} C_\alpha \prod_{l=0}^{k-1} (l - 2/\alpha) + \Theta(\kappa^2) \quad (53)$$

so that the outage probability is given by

$$\begin{aligned} \mathbf{P}(SIR \leq \beta) &= \lambda \zeta^{\frac{2}{\alpha}} C_\alpha \sum_n \sum_k a_{nk} n^{\frac{2}{\alpha}-k} \prod_{l=0}^{k-1} (l - 2/\alpha) \\ &\quad + \Theta(\kappa^2) \\ &= \lambda \zeta^{\frac{2}{\alpha}} \frac{C_\alpha}{K_\alpha} + \Theta(\kappa^2) = \epsilon \end{aligned} \quad (54)$$

and with  $\zeta = \beta R^\alpha$  and  $K_\alpha$  as in (12), solving for  $\lambda$  yields the result.

For completeness, we conclude by demonstrating how thermal noise can be included in the analysis. To include noise,  $I_\Phi$  must be replaced by  $I_\Phi + \frac{1}{\rho} N_o$  and so the transform of the distribution  $f_{I_\Phi + \frac{1}{\rho} N_o}(x)$ , given by  $\mathcal{L}_{I_\Phi + \frac{1}{\rho} N_o}(\zeta) =$

$\mathcal{L}_{I_\Phi}(\zeta) \mathcal{L}_{N_o}(\zeta/\rho)$ , replaces  $\mathcal{L}_{I_\Phi}(\zeta)$  in the above derivations. This follows from the property of Laplace transforms that the transform of the sum of independent variables is the product of the transforms. Now the transform of the noise is  $\mathcal{L}_{N_o}(\zeta/\rho) = e^{\zeta N_o/\rho}$ . Furthermore, since  $\frac{d}{d\zeta} \mathcal{L}_{N_o}(\zeta) = \frac{N_o}{\rho} \mathcal{L}_{N_o}(\zeta)$  we have

$$\frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi + N}(\zeta) = \mathcal{L}_N(\zeta) \left( \sum_{k=0}^p \binom{p}{k} \left( \frac{N_o}{\rho} \right)^{p-k} \frac{d^k}{d\zeta^k} \mathcal{L}_{I_\Phi}(\zeta) \right) \quad (55)$$

The expression for  $\frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi + \frac{1}{\rho} N_o}(\zeta)$  now replaces  $\frac{d^p}{d\zeta^p} \mathcal{L}_{I_\Phi}(\zeta)$  in (10). Under small outage constraints, the first order Taylor expansion of the probability of outage can be made in a manner analogous to equations (52) through (54) leading to:

$$K_\alpha = \left[ \sum_{n \in \mathcal{N}} \sum_{j=0}^k \binom{k}{j} \left( \frac{N_o}{\rho} \right)^{k-j} a_{nk} n^{\frac{2}{\alpha}-j} \prod_{l=0}^{j-1} (l - 2/\alpha) \right]^{-1}. \quad (56)$$

Note that this expansion is only valid when outage due to the fading of the intended signal and thermal noise is less than  $\epsilon$  in the absence of any interference.

## APPENDIX B PROOF OF PROPOSITION 1

To demonstrate the above let the interfering signals and the desired signal be Nakagami fading with different parameters  $m_i$  and  $m_o$  respectively. The CCDF of the received power is:  $F_{S_0^c}^c(s) = e^{-m_o s} \sum_{k=0}^{m_o-1} \frac{(m_o s)^k}{k!}$  with  $m_o = 1$  being the Rayleigh case. According to Theorem 1:

$$\mathbf{P}(SIR \geq \beta) = \sum_{k=0}^{m_o-1} \frac{(-\zeta)^k}{k!} \frac{d^k}{d\zeta^k} \mathcal{L}_{I_\Phi}(\zeta) \Big|_{\zeta=m_o \beta R^\alpha} \quad (57)$$

Note that  $\zeta$  now includes the fading parameter  $m_o$ . To determine the Laplace transform of the shot noise process, with  $m_i$  denoting the Nakagami parameter for all interfering transmissions the MGF of each mark is altered to be

$$E \left[ e^{-\zeta S_i |x|^{-\alpha}} \right] = \frac{1}{(1 + \zeta/m_i |x|^\alpha)^{m_i}} \quad (58)$$

and the integral in (5) can be evaluated as [37]

$$\begin{aligned} \mathcal{L}_{I_\Phi}(\zeta) &= \exp \left\{ -2\pi\lambda \int_0^\infty u \left( 1 - \frac{1}{(1 + u^\alpha/\zeta)^{m_i}} \right) du \right\} \\ &= \exp \left\{ -2\pi\lambda \int_0^\infty \frac{\sum_{k=1}^{m_i} \binom{m_i}{k} u^{k\alpha+1} (m_i/\zeta)^k}{(1 + u^\alpha m_i/\zeta)^{m_i}} du \right\} \\ &= \exp \left\{ -\lambda C_{\alpha, m_i} \left( \frac{\zeta}{m_i} \right)^{\frac{2}{\alpha}} \right\} \end{aligned} \quad (59)$$

where  $C_{\alpha, m_i} = C_{\alpha, m}$  in (15). By Theorem 1, the optimal contention density is:

$$\bar{\lambda}_\epsilon = \frac{m_i^{\frac{2}{\alpha}} K_{\alpha, m_o} \epsilon}{m_o^{\frac{2}{\alpha}} C_{\alpha, m_i} \beta^{\frac{2}{\alpha}} R^2} \quad (60)$$

where  $K_{\alpha, m_o}$  is given by (12). If  $m_o$  is set to 1 (Rayleigh fading) with  $K_{\alpha, 1} = 1$ , and  $m_i \rightarrow \infty$ , the MGF of the power

fading mark on each interferer approaches  $e^{-\zeta|x|^{-\alpha}}$ . Hence,

$$\begin{aligned}\mathcal{L}_{I_\Phi}(\zeta) &= \exp\left\{-2\pi\lambda\int_0^\infty x(1-e^{-\zeta|x|^{-\alpha}})dx\right\} \\ &= \exp\left\{-\pi\lambda\zeta\frac{2}{\alpha}\Gamma(1-2/\alpha)\right\}\end{aligned}\quad (61)$$

indicating that

$$\lim_{m\rightarrow\infty}\frac{C_{\alpha,m}}{m^{\frac{2}{\alpha}}} = \pi\Gamma(1-2/\alpha).\quad (62)$$

If in addition  $m_i = \infty$  with  $C_{\alpha,\infty} = \pi\Gamma(1-2/\alpha)$  and  $m_o$  is allowed to approach infinity, the distribution of  $S_0$  becomes an impulse at  $S_0 = 1$ . Weber, *et al.* [11], derived bounds on the optimal contention density for path loss only (non-fading):

$$\left(\frac{\alpha-1}{\alpha}\right)\frac{\epsilon}{\pi\beta^{\frac{2}{\alpha}}R^2} \leq \bar{\lambda}_\epsilon \leq \frac{\epsilon}{\pi\beta^{\frac{2}{\alpha}}R^2}.$$

This gives

$$\left(\frac{\alpha-1}{\alpha}\right)\frac{1}{\pi} \leq \lim_{m\rightarrow\infty}\frac{K_{\alpha,m}}{C_{\alpha,m}} \leq \frac{1}{\pi}\quad (63)$$

which for fixed  $C_{\alpha,\infty}$  determines the asymptotic orderwise increase of  $K_{\alpha,m}$ :  $\lim_{m\rightarrow\infty}\frac{K_{\alpha,m}}{m^{\frac{2}{\alpha}}} = c_1$ , for some finite, nonzero constant  $c_1$ . To fully demonstrate the orderwise behavior of  $K_{\alpha,m}$ , the bounds

$$1 \leq \frac{K_{\alpha,m}}{m^{\frac{2}{\alpha}}} \leq c_1\quad (64)$$

hold since  $\frac{K_{\alpha,m}}{m^{\frac{2}{\alpha}}}$  is monotonically increasing but approaches the limit  $c_1$ .

Equation (60) is more general than (13) for which  $m_o = m_i = m$ , but while the physical significance of modeling this disparity between desired and interfering statistics is dubious<sup>1</sup>, it allows the behavior of  $K_{\alpha,m}$  and  $C_{\alpha,m}$  to be studied. It was shown in [11] that the upper bound is fairly tight which implies that in fact  $\lim_{m\rightarrow\infty}\frac{K_{\alpha,m}}{C_{\alpha,m}} \approx \frac{1}{\pi}$ . While the upper bound holds, we have numerically that the ratio  $\frac{K_{\alpha,m}}{C_{\alpha,m}}$  does in fact approach the upper bound with increasing  $m$ . Approximating closely the limit  $\frac{K_{\alpha,\infty}}{C_{\alpha,\infty}}$  as  $\frac{1}{\pi}$ , we can approximate  $c_1$  very closely as  $c_1 \approx \Gamma(1-2/\alpha)$ .

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<sup>1</sup>For moderately dense ad hoc or sensor networks with narrowband transmission, the key interferers are the nearest ones, and it is unlikely that the propagation statistics of the interfering and desired signals would be widely different.

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