On the Capacity and Diversity-Multiplexing Tradeoff of the Two-Way Relay Channel

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Abstract

This paper considers a multiple input multiple output (MIMO) two-way relay channel, where two nodes want to exchange data with each other using multiple relays. An iterative algorithm is proposed to achieve the optimal achievable rate region, when each relay employs an amplify and forward (AF) strategy. The iterative algorithm solves a power minimization problem at every step, subject to minimum signal-to-interference-and-noise ratio constraints, which is non-convex, however, for which the Karush Kuhn Tuker conditions are sufficient for optimality. The optimal AF strategy assumes global channel state information (CSI) at each relay. To simplify the CSI requirements, a simple amplify and forward strategy, called dual channel matching, is also proposed, that requires only local channel state information, and whose achievable rate region is close to that of the optimal AF strategy. In the asymptotic regime of large number of relays, we show that the achievable rate region of the dual channel matching and an upper bound differ by only a constant term and establish the capacity scaling law of the two-way relay channel. Relay strategies achieving optimal diversity-multiplexing tradeoff are also considered with a single relay node. A compress and forward strategy is shown to be optimal for achieving diversity multiplexing tradeoff for the full-duplex case, in general, and for the half-duplex case in some cases.

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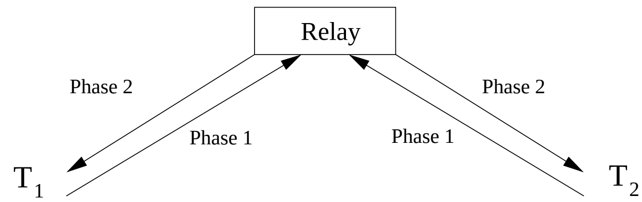

Fig. 1. Two way relay channel communication protocol

I. Introduction

We consider a multiple antenna two-way relay channel as shown in Fig. 1, where two nodes T_1 and T_2 want to exchange information with each other with the help of a relay node and all the nodes are equipped with one or more than one antenna. The two-way relay channel models the communication scenario where the destination terminal also has some data to send to source terminal e.g. downlink and uplink in cellular communication, or packet acknowledgments in a wireless network. The general discrete memoryless two-way relay channel was introduced in [1], and the multiple antenna two-way relay channel in [2]. In the literature, the two-way relay channel is also known by several other names, including the: bidirectional relay channel [3]–[5] and analog network coding [6].

A specific embodiment of a multiple antenna two-way relay channel that assumes half-duplex relays and the absence of a direct path between source and destination was proposed in [2]. An illustration is provided in Fig. 1. As shown in Fig. 1, in phase 1 or the first time slot, both terminals T_1 and T_2 are scheduled to transmit simultaneously while the relay receives. In phase 2 or the second time slot, the relay is scheduled to transmit while terminals T_1 and T_2 receive. The key idea with the two-way relay channel is that each terminal can cancel the interference (generated by its own transmission) from the signal it receives from the relay to recover the transmission from the other terminal. The idea is reminiscent of work in network coding [7], though note that here the coding is done in the analog domain, [6] rather than in digital domain [7]. In this paper we only consider multiple antenna two-way relay channel and for brevity, drop the prefix multiple antenna from here onwards.

There has been a growing interest in finding the capacity region of the two-way relay channel with a single relay node [3]–[5], [8]–[13]. Achievable sum rate expressions (sum of the rates achievable from $T_1 \rightarrow T_2$ and $T_2 \rightarrow T_1$ links) have been derived in [2] and [3]–[5], for the half-duplex two-way relay channel, using amplify and forward (AF), decode and forward (DF) and compress and forward (CF) at the relay. It is shown that in a two-way relay channel, it is possible to remove the $\frac{1}{2}$ rate loss factor in spectral efficiency due to the half duplex assumption on the nodes. For a general full-duplex two-way relay channel with a single relay node (T_1, T_2 and relay can transmit and receive at the same time) achievable rate regions are derived in [8] for AF, DF, and CF. For the AWGN two-way relay channel (no fading), using nested lattice coding and DF at the relay, the achievable rate region has been shown to be very close to the upper bound for all SNRs [10], [11]. Using the deterministic channel approach, the achievable rate region has been shown to be at most three bits away from the upper bound for the full-duplex two-way relay channel [9]. The capacity region of the two-way relay channel has also been studied in [12], [13], where in [13], it has been shown that in the low SNR regime the upper bound can be achieved by choosing a suitable relay mapping function, together with LDPC codes. The achievable rate region [3]–[5], [8]–[13] does not meet the upper bound [14], in general. Consequently, the problem of finding the capacity region of the two-way relay channel is currently open.

The problem of finding the capacity region of the two-way relay channel becomes even more challenging when there are multiple relay nodes that can help T_1 and T_2, and to the best of our knowledge has not been addressed in the literature. The problem becomes hard, because it is known that for the one-way relay channel with multiple relay nodes, DF does not work well [15], while the partial DF and distributed CF [15] lead to complicated achievable rate regions that are very hard to compute and analyze. The same conclusion holds true for the two-way relay channel; the only simple strategy that is well suited for multiple relay nodes is AF. With this motivation, in this paper we attempt to find the optimal relay beamformers that maximize the achievable rate region of the two-way relay channel with AF. For the one-way relay channel with multiple relays, optimal relay beamformers have been found [16], however, they are not known for the two-way relay channel.

For the case when both T_1 and T_2 have a single antenna, and each relay has an arbitrary number of antennas, we solve the problem of finding optimal relay beamformers by recasting it as an iterative power minimization algorithm. The iterative algorithm, at each step, solves a power minimization problem with minimum signal-to-interference-noise (SINR) constraints, for which satisfying the Karush Kuhn Tucker (KKT) conditions [17], [18] are sufficient for optimality. We consider both the sum power constraint across relays, as well as an individual relay power constraint. The optimal AF solution requires each relay to have channel state information (CSI) for all relays and leads to an achievable rate region that cannot be expressed in closed form.

For the case when each relay knows its own CSI, finding the optimal AF strategy is quite hard and intractable, even for the one-way relay channel case [16]. To remove the global CSI requirement, and to obtain a simple achievable rate region expression, next, we propose a simple AF strategy, called dual channel matching strategy, which works for any number of antennas at T_1 and T_2. In dual channel matching, relay k transmits the received signal multiplied with $(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*})$, if the channel between T_1 and relay k is $\mathbf{H}_k$, between relay k and T_2 is $\mathbf{G}_k^r$, between T_2 and relay k is $\mathbf{G}_k$ and between relay k and T_1 is $\mathbf{H}_k^r$. Using dual channel matching, we lower bound the achievable rate region of the optimal AF strategy, which is unknown for more than one antenna at T_1 and T_2, and bound the gap between the optimal AF strategy and the upper bound. The dual channel matching is quite simple to implement and its achievable rate region can be shown to be quite close (by simulation) to the optimal AF strategy, when T_1 and T_2 each have single antenna.

We upper bound the capacity region of the two-way relay channel using the cut-set bound [19] on the broadcast cut T_1 (T_2), and $r_1, r_2, \dots, r_K$, and the multiple access cut $r_1, r_2, \dots, r_K$ and T_2 (T_1), over all possible two phase protocols (with different time allocation between first and second phase). We show that the gap between the upper and lower bound (dual channel matching) is quite small for small values of K. In the limit $K \to \infty$, we show that the gap is constant with increasing K, and thus establish the scaling law [20] of the capacity region of the two-way relay channel, which shows that $\frac{M}{2}\log K$ bits can be transmitted from both $T_1 \to T_2$ and $T_2 \to T_1$, simultaneously.

We also consider the problem of finding relay transmission strategies to achieve the optimal diversity multiplexing (DM)-tradeoff [21] of the two-way relay channel with a single relay node, in the presence of a direct path between T_1 and T_2. The DM-tradeoff captures the maximum rate of fall of error probability with signal to noise ratio (SNR), when rate of transmission is increased as $r \log \mathsf{SNR}$. The DM-tradeoff for the two-way relay channel is a two-dimensional region spanned by the $(d_{12}(r_{12}, r_{21}), d_{21}(r_{12}, r_{21}))$, where d_{12} and d_{21} are the negatives of the exponent of the probability of error from $T_1 \to T_2$ and $T_2 \to T_1$, respectively, when T_1 is transmitting at rate $r_{12} \log \mathsf{SNR}$ and T_2 at $r_{21} \log \mathsf{SNR}$. The DM-tradeoff for the one-way relay channel has been studied in [22]–[25], where notably in [25], it has been shown that the CF strategy achieves the DM-tradeoff for both the full-duplex as well as the half-duplex case. The DM-tradeoff of the two-way relay channel has been recently studied in [26], where upper and lower bounds are obtained on the DM-tradeoff which are shown to match for the case when each node has a single antenna.

We first consider the full-duplex two-way relay channel and show that a slightly modified version of the CF strategy [27] achieves the optimal DM-tradeoff. More importantly, we show that $d_{12}(r_{12}, r_{21})$ ($d_{21}(r_{12}, r_{21})$) does not depend on r_{21} (r_{12}) and the two-way relay channel can be decoupled into two one-way relay channels using the CF strategy. Then we consider the more interesting case of half-duplex nodes, where the achievable rate regions are protocol dependent. For the two-way relay channel it is not known which protocol achieves the highest possible rates [3]–[5]. We use a three phase protocol, where in phase one T_1 transmits to both the relay and T_2, in phase two T_2 transmits to both the relay and T_1 and in phase three the relay transmits to T_1 and T_2. This three phase protocol makes use all the direct links between different nodes in a two-way relay channel. For this three phase protocol, we propose a modified CF strategy and show that it can achieve the optimal DM-tradeoff in some cases. We conjecture that our strategy can also achieve the optimal DM-tradeoff in general, but we are yet to prove it.

Notation: The following notation is used in this paper. The superscripts $^T,^*$ represent the transpose and transpose conjugate. $\mathbf{M}$ denotes a matrix, $\mathbf{m}$ a vector and m_i the i^{th} element of $\mathbf{m}$. For a matrix $\mathbf{M} = [\mathbf{m}_1 \ \mathbf{m}_2 \ \ldots \ \mathbf{m}_n]$ by $vec(\mathbf{M})$ we mean $[\mathbf{m}_1^T \ \mathbf{m}_2^T \ \ldots \ \mathbf{m}_n^T]^T$. $det(\mathbf{A})$ and $tr(\mathbf{A})$ denotes the determinant and trace of matrix $\mathbf{A}$, respectively. $\mathbb{E}$ denotes the expectation. $||\cdot||$ denotes the usual Euclidean norm of a vector and $|\cdot|$ denotes the absolute value of a scalar. $\mathbf{I}_m$ is a $m \times m$ identity matrix. $|\mathcal{X}|$ is the cardinality of set $\mathcal{X}$. We use the usual notation for $u(x) = \mathcal{O}(v(x))$ if $|\frac{u(x)}{v(x)}|$ remains bounded, as $x \rightarrow \infty$. $x \sim \mathcal{CN}(0, \sigma)$ means x is a circularly symmetric complex Gaussian random variable with zero mean and variance σ and $x|y \sim \mathcal{CN}(0, \sigma)$ means given y, x is a circularly symmetric complex Gaussian random variable with zero mean and variance σ. $\mathbb{C}^{MN}$ denotes the set of $M \times N$ matrices with complex entries. $x_n \xrightarrow{w.p.1} y$ denotes that the sequence of random variables x_n converge to a random variable y with probability 1. We use $a \underset{w.p.1}{=} b$ to denote equality with probability 1 i.e. $Prob.(a = b) = 1$ and $\underset{w.p.1}{\leq}$ is defined similarly. $I(x; y)$ denotes the mutual information between x and y and $h(x)$ the differential entropy of x [19]. To define a variable we use the symbol $:=$.

Organization: The rest of the paper is organized as follows. In Section II, we describe the two-way relay channel system model, the protocol under consideration and the key assumptions. In Section III, we obtain the optimal AF strategy to maximize the achievable rate region of the two-way relay channel. In Section IV, we introduce a simple AF strategy, dual channel matching, and lower bound the achievable rate region of the optimal AF strategy of Section III. In Section V, we derive an upper bound on the capacity of the two-way relay channel capacity and compare it with the achievable rate region of the optimal AF strategy and dual channel matching. In Section VI, we show that the CF strategy can achieve the optimal DM-tradeoff for full-duplex two-way relay channel, in general, and in some cases for the half-duplex case. Final conclusions are made in Section VII.

II. System and Channel Model

In this section we describe the two-way relay channel system model under consideration, and then present the relevant signal and channel models.

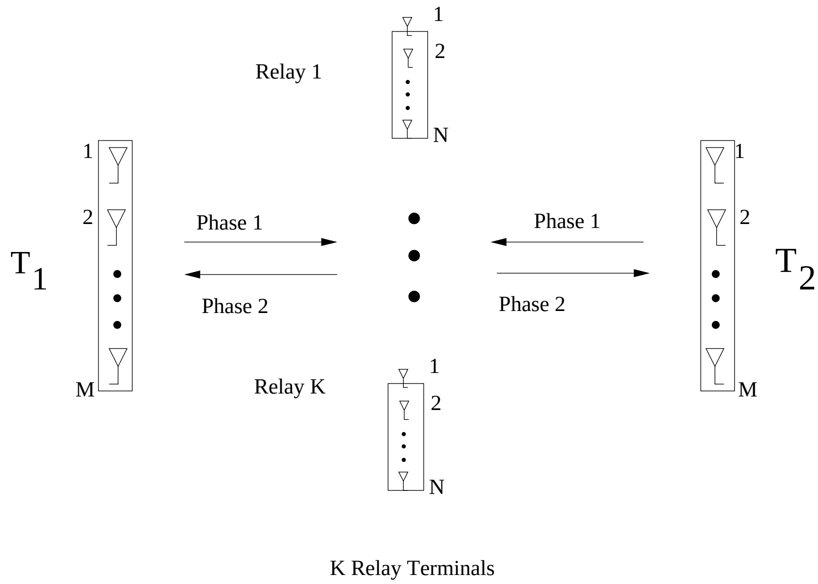

Fig. 2. Two-way relay channel system model with two phase communication

A. System Model

For the first part of the paper Section III, IV, and V, we consider a wireless network where there are two terminals T_1 and T_2 who want to exchange information via K relays, as shown in Fig. 2. The K relays do not have any data of their own and only help T_1 and T_2 communicate. The K relays are assumed to be located randomly and independently so that the channel coefficients between each relay and T_1 and T_2 are independent. We also assume that there is no direct path between T_1 and T_2 and that they can communicate only through the K relays. This is a realistic assumption when relaying is used for coverage improvement in cellular systems, since at the cell edge the signal to noise ratio is extremely low for the direct path. In ad-hoc networks, it can be the case that two terminals want to communicate, but are out of each other's transmission range.

We assume that both the terminals T_1 and T_2 have M antennas and all the K relays have N antennas each. We further assume that both the terminals and all the relays can operate only in half-duplex mode (cannot transmit and receive at the same time). The communication protocol is summarized as follows [2]. In any given time slot, for the first α fraction of time, called the *transmit phase*, both T_1 and T_2 are scheduled to transmit and all the relays receive a superposition of the signals transmitted from T_1 and T_2. In the rest $(1-\alpha)$ fraction of the time slot, called the *receive phase*, all the relays are scheduled to transmit simultaneously and both the terminals receive. Both T_1 and T_2 are assumed to have power constraint of P, while for relays we assume two different power constraints, the sum power constraint

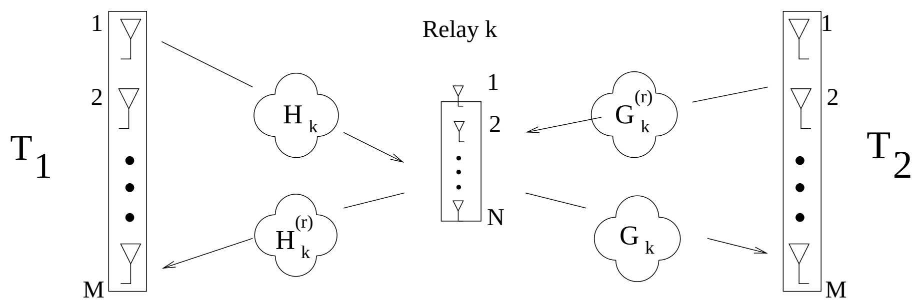

Fig. 3. Channel model for the two-way relay channel between T_1, T_2 and relay k

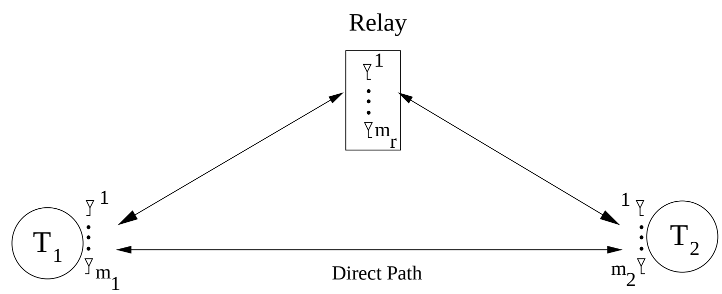

Fig. 4. System Model of the two-way relay channel with direct path for Section VI

where the sum of the power of all relays is $\leq P_R$ or the individual power constraint where each relay has power constraint of P_R.

For the second part of the paper, Section VI, we assume a two-way relay channel with a single relay node and the presence of a direct path between T_1 and T_2 as shown in Fig. 4. We assume that T_1 has m_1 antennas, T_2 has m_2 antennas, and the relay node has m_r antennas.

B. Channel and Signal Model

Throughout this paper we assume that all the channels are frequency flat slow fading block fading channels, where in a block of time duration T_c (called the coherence time), the channel coefficients remain constant and change independently from block to block. We assume that T_c is more that the duration of time slot used by T_1 and T_2 to communicate with each other as described before. As shown in Fig. 3, let the forward channel between T_1 and the k^{th} relay be $\mathbf{H}_k = [\mathbf{h}_{1k} \ \mathbf{h}_{2k} \ \ldots \ \mathbf{h}_{Mk}]$ and the backward channel between k^{th} relay and T_1 be $\mathbf{H}_k^r = [\mathbf{h}_{k1}^r \ \mathbf{h}_{k2}^r \ \ldots \ \mathbf{h}_{kM}^r]$. Similarly let the forward channel between k^{th} relay and T_2 be $\mathbf{G}_k = [\mathbf{g}_{k1} \ \mathbf{g}_{k2} \ \ldots \ \mathbf{g}_{kM}]$ and the backward channel between T_2

and the k^{th} relay be $\mathbf{G}_k^r = [\mathbf{g}_{1k}^r \ \mathbf{g}_{2k}^r \ \dots \ \mathbf{g}_{Mk}^r]$. For Section VI, where the direct path between T_1 and T_2 is considered, the channel between T_1 and T_2 is denoted by $\mathbf{H}_{12}$ and in the reverse direction by $\mathbf{H}_{12}^r$. We assume that $\mathbf{H}_k, \mathbf{G}_k^r \in \mathbb{C}^{N\times M}, \mathbf{H}_k^r, \mathbf{G}_k \in \mathbb{C}^{M\times N}, \mathbf{H}_{12} \in \mathbb{C}^{m_2\times m_1}, \mathbf{H}_{12}^r \in \mathbb{C}^{m_1\times m_2}$ with independent and identically distributed (i.i.d.) $\mathcal{CN}(0,1)$ entries.

For the first part of the paper Section III, IV and V, we consider the following signal model. The $N \times 1$ received signal at the k^{th} relay is given by

$$\mathbf{r}_k = \sqrt{\frac{P}{M}}\mathbf{H}_k\mathbf{x}_1 + \sqrt{\frac{P}{M}}\mathbf{G}_k^r\mathbf{x}_2 + \mathbf{n}_k \tag{1}$$

if $\mathbf{x}_1$ and $\mathbf{x}_2$ are the $M \times 1$ signals transmitted from T_1 and T_2 to be decoded at T_2 and T_1 respectively, with $\mathbb{E}\{\mathbf{x}_1{}^*\mathbf{x}_1\} = \mathbb{E}\{\mathbf{x}_2{}^*\mathbf{x}_2\} = M$, P is the power transmitted by T_1 and T_2, respectively. The noise $\mathbf{n}_k$ is the $N\times 1$ spatio-temporal white complex Gaussian noise independent across relays with $\mathbb{E}(\mathbf{n}_k\mathbf{n}_k^*) = \ \mathbf{I}_N$. Relay k processes its incoming signal to transmit a $N \times 1$ signal $\mathbf{t}_k = \mathbf{W}_k\mathbf{r}_k$ with $\sum_{k=1}^K \mathbb{E}\{\mathbf{t}_k^*\mathbf{t}_k\} \le P_R$ (sum power constraint) or $\mathbb{E}\{\mathbf{t}_k^*\mathbf{t}_k\} \le P_R$ (individual power constraint) in the receive phase. The $M \times 1$ received signals $\mathbf{y}_1$ and $\mathbf{y}_2$ at terminal T_1 and T_2, respectively, in the receive phase, are given by

$$\mathbf{y}_1 = \sum_{k=1}^K \mathbf{H}_k^r\mathbf{t}_k + \mathbf{z}_1, \tag{2}$$

$$\mathbf{y}_2 = \sum_{k=1}^K \mathbf{G}_k\mathbf{t}_k + \mathbf{z}_2, \tag{3}$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ are $M \times 1$ spatio-temporal white complex Gaussian noise vectors with $\mathbb{E}(\mathbf{z}_1\mathbf{z}_1^*) = \mathbb{E}(\mathbf{z}_2\mathbf{z}_2^*) = \ \mathbf{I}_M$.

Throughout this paper we assume that both T_1 and T_2 perfectly know $\{\mathbf{H}_k, \mathbf{H}_k^r, \mathbf{G}_k, \mathbf{G}_k^r\}\ \forall\ k,\ k = 1, 2, , \dots K$ in the receive mode. To be precise, in the receive phase (i.e. when T_1 and T_2 receive signal from all the relays), T_1 and T_2 both know $\{\mathbf{H}_k, \mathbf{G}_k\}$ and $\{\mathbf{H}_k^r, \mathbf{G}_k^r\}\ \forall\ k,\ \ k = 1, 2, , \dots K$. We also assume that no transmit CSI is available at T_1 and T_2, i.e. in the transmit phase T_1 and T_2 have no information about what the realization of $\mathbf{H}_k$ and $\mathbf{G}_k$ is going to be when it transmits its signal to all the relays in the transmit phase, respectively.

In this paper we assume different CSI assumptions at the relay. For finding the optimal AF strategy (Section III) we assume that each relay knows $\mathbf{H}_k, \mathbf{G}_k^r, \mathbf{G}_k, \mathbf{H}_k^r$ for all $k = 1, 2 \dots, K$. To reduce the CSI requirements next, we present a simple AF strategy in Section IV where we assume that relay k only knows $\mathbf{H}_k, \mathbf{G}_k^r, \mathbf{G}_k, \mathbf{H}_k^r$. In Section VI, we assume that the relay knows $\mathbf{H}_1, \mathbf{G}_1^r, \mathbf{G}_1, \mathbf{H}_1^r$, as well as $\mathbf{H}_{12}$, the channel coefficient between T_1 and T_2.

III. Optimal AF strategy for two-way relay channel

In this section we will find optimal relay beamformers that maximize the achievable rate region of the two-way relay channel with AF, when T_1 and T_2 have a single antenna each, $M = 1$. For simplicity of exposition, in this section we consider the case when each relay nodes has a single antenna, $N = 1$. Generalizations to $N > 1$ are straightforward, and will be described later.

To start with, because of single antenna restriction, the channel between T_1 and relay k is denoted by h_k and between relay k and T_2 denoted by g_k. For the reverse direction the channel coefficients are the same as in forward direction but with an added superscript r, e.g. channel coefficient between relay k and T_1 is denoted by h_k^r. With AF strategy, each relay node transmits the received signal multiplied with w_k to both T_1 and T_2. Thus, if x_1 and x_2 is the transmitted signal from T_1 and T_2, respectively, then the received signal at T_1, y_1, and T_2, y_2 is

$$\begin{aligned} y_1 &= \sum_{k=1}^{K} \sqrt{P} h_k^r w_k g_k^r x_2 + \sqrt{P} h_k^r w_k h_k x_1 + h_k^r w_k n_k + z_1, \\ y_2 &= \sum_{k=1}^{K} \sqrt{P} g_k w_k h_k x_1 + \sqrt{P} g_k w_k g_k^r x_1 + g_k w_k n_k + z_2, \end{aligned} \tag{4}$$

where $n_k, \; \forall \; k = 1, \ldots, K$ is $\mathcal{CN}(0,1)$ noise added at relay k and z_1, and z_2 are $\mathcal{CN}(0,1)$ added at T_1 and T_2. Since x_1 and x_2 are known at T_1 and T_2, respectively, their contribution can be removed from the received signal at T_1 and T_2, respectively. Let the rate of transmission from T_1 to T_2 be R_{12} and from T_2 to T_1 be R_{21}, then from (4)

$$\begin{aligned} R_{12} &= \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} g_k w_k h_k\right|^2\right)}{1 + \sum_{k=1}^{K} |g_k w_k|^2}\right), \\ R_{21} &= \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} h_k^r w_k g_k^r\right|^2\right)}{1 + \sum_{k=1}^{K} |h_k^r w_k|^2}\right). \end{aligned}$$

Thus, the achievable rate region for the two-way relay channel with AF for a sum power constraint across all relays, i.e. $p_R = P\sum_{k=1}^{K}(|w_k h_k|^2 + |w_k g_k^r|^2) + \sum_{k=1}^{K} |w_k|^2 \le P_R$ is the set $\mathcal{R}(P, P_R) = \cup_{p_R \le P_R}(R_{12}, R_{21})$ and for individual power constraint at each relay, i.e. $p_{kR} = P(|w_k h_k|^2 + |w_k g_k^r|^2) + |w_k|^2 \le P_R$ is the set $\mathcal{R}(P, P_R) = \cup_{p_{kR} \le P_R, \; k=1,\ldots,K}(R_{12}, R_{21})$. Therefore, the problem is to find optimal w_k's that achieve the boundary points of the region $\mathcal{R}(P, P_R)$, for both the sum power constraint and an individual power constraint.

For the one-way relay channel, no communication from T_2 to T_1, optimal w_k's have been found in [16] to maximize R_{12}. The solution of [16], provides an upper bound on individual rates R_{12} and R_{21} and is equivalent to solutions where R_{12} or R_{21} is greedily maximized disregarding the other. The problem in the two-way relay channel case is to find optimal w_k's such that $R_{sum} = R_{12} + R_{21}$ is maximized, for each $\beta \in [0, 1]$, where $R_{12} = \beta R_{sum}$, and $R_{21} = (1 - \beta) R_{sum}$. Towards that end, we use the rate profile method [28] to identify w_k's that meet the boundary point of $\mathcal{R}(P, P_R)$. Next, we only consider the sum power constraint across the relays. For individual power constraints the same procedure can be applied as pointed out later. Thus, the optimization problem can be formulated as follows.

$$\begin{array}{ll} \text{Maximize}_{w_k,\ k=1,2,\ldots,K} & R_{sum} \\ \text{subject to} & \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} g_k w_k h_k\right|^2\right)}{1+\sum_{k=1}^{K} |g_k w_k|^2}\right) \geq \beta R_{sum}, \\ & \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} h_k^r w_k g_k^r\right|^2\right)}{1+\sum_{k=1}^{K} |h_k^r w_k|^2}\right) \geq (1-\beta) R_{sum}, \\ & P\sum_{k=1}^{K}(|w_k h_k|^2 + |w_k g_k^r|^2) + \sum_{k=1}^{K} |w_k|^2 \leq P_R. \end{array} \tag{5}$$

An equivalent problem to this problem is the following iterative power minimization problem subject to rate constraints,

$$\begin{array}{ll} \text{Minimize}_{w_k,\ k=1,2\ldots,K} & p_R = P\sum_{k=1}^{K}(|w_k h_k|^2 + |w_k g_k^r|^2) + \sum_{k=1}^{K} |w_k|^2 \\ \text{subject to} & \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} g_k w_k h_k\right|^2\right)}{1+\sum_{k=1}^{K} |g_k w_k|^2}\right) \geq \beta R^u_{sum}, \\ & \log\left(1 + \frac{P\left(\left|\sum_{k=1}^{K} h_k^r w_k g_k^r\right|^2\right)}{1+\sum_{k=1}^{K} |h_k^r w_k|^2}\right) \geq (1-\beta) R^u_{sum}, \end{array} \tag{6}$$

where at each iteration R^u_{sum} is changed to maximize the achievable rate, subject to power constraint. To be precise, if the value of R^u_{sum} at iteration i is say x and the solution to (6) is feasible (i.e. if $p_R \leq P_R$) [1], then x is incremented in next iteration, otherwise decreased. Choice of the step size of increase or decrease determines the speed of convergence to the optimal rate R^u_{sum}, for which $p_R \leq P_R$. One possible starting point for R^u_{sum} is 2 times the maximum R_{12} provided by [16] for one way relay channel. The step size can be chosen by bisection between the last feasible R^u_{sum} (initially 0) and the last infeasible R^u_{sum}. Even though this equivalent problem provides a solution to (5) in a iterative manner, the problem (6) is in general non-convex, and not easy to solve. To overcome this limitation, we recast the problem (6) as a standard power minimization problem subject to signal-to-interference-noise ratio (SINR) [18], where the forwarded noise from each relay plays the role of interference. For a given β

[1]For an individual power constraint the same can be done by checking at each iteration whether the obtained solution p_R is feasible with individual power constraints or not.

and R^u_{sum}, the problem (6) is of the form

$$\begin{array}{ll} \text{Minimize}_{w_k,\ k=1,2\dots,K} & p_R = P\sum_{k=1}^K(|w_kh_k|^2+|w_kg^r_k|^2)+\sum_{k=1}^K|w_k|^2 \\ \text{subject to} & \frac{\left|\sum_{k=1}^K g_kw_kh_k\right|^2}{1+\sum_{k=1}^K|g_kw_k|^2} \geq \frac{2^{\beta R^u_{sum}}-1}{P} := \gamma_0 \\ & \frac{\left|\sum_{k=1}^K h^r_kw_kg^r_k\right|^2}{1+\sum_{k=1}^K|h^r_kw_k|^2} \geq \frac{2^{(1-\beta)R^u_{sum}}-1}{P} := \gamma_1. \end{array} \quad (7)$$

This problem again is non-convex, however, it is of the form

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & ||a_i(x)||^2-|b_i(x)|^2 \leq 0,\ \forall\ i, \end{array} \quad (8)$$

where $f(x)$ is a convex function, $a_i(x)$ is an affine function of x and $b_i(x)\geq 0\ \ \forall\ i$, by noting that if $\sum_{k=1}^K g_kw_kh_k$, or $\sum_{k=1}^K h^r_kw_kg^r_k$ are less than zero or complex, then they can be scaled by appropriate phases to make them real and positive, without changing the objective function or the constraints [2].

For the problem (8), it has been shown in [18], that if the problem is strictly feasible, then KKT conditions [17] are necessary and sufficient to find the optimal solution. It is easy to see that the problem (7) is strictly feasible and therefore KKT conditions are sufficient for optimality. The Lagrangian of problem (7) is of the form

$$\mathcal{L} = \mathbf{wAw}^*+\lambda_1\left(\mathbf{wBw}^*-\frac{1}{\gamma_0}|\mathbf{cw}^T|^2+1\right)+\lambda_2\left(\mathbf{wDw}^*-\frac{1}{\gamma_1}|\mathbf{ew}^T|^2+1\right),$$

where $\mathbf{w}=[w_1\ \dots\ w_K]$ and

$$\mathbf{A}=\begin{bmatrix} P(|h_1|^2+|g^r_1|^2)+1 & 0 & \dots & 0 \\ 0 & P(|h_2|^2+|g^r_2|^2)+1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & P(|h_K|^2+|g^r_K|^2)+1 \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix} |g_1|^2 & 0 & \dots & 0 \\ 0 & |g_2|^2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & |g_K|^2 \end{bmatrix},\ \mathbf{D}=\begin{bmatrix} |h^r_1|^2 & 0 & \dots & 0 \\ 0 & |h^r_2|^2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & |h^r_K|^2 \end{bmatrix},$$

and $\mathbf{c}=\begin{bmatrix} g_1h_1 & \dots & g_Kh_K \end{bmatrix}$, $\mathbf{e}=\begin{bmatrix} h^r_1g^r_1 & \dots & h^r_Kg^r_K \end{bmatrix}$.

Differentiating the Lagrangian yields

$$\left(\mathbf{A}+\lambda_1\mathbf{B}+\lambda_2\mathbf{D}-\frac{\lambda_1}{\gamma_0}\mathbf{c}^*\mathbf{c}+\frac{\lambda_2}{\gamma_1}\mathbf{e}^*\mathbf{e}\right)\mathbf{w}=0,$$

[2] An immediate consequence of this property is that the optimal solution does not change if all g'_ks are scaled by $e^{j\phi_1}$, or all h^r_k's are scaled by $e^{j\phi_2}$.

and the optimal $\mathbf{w}$ is found by solving for λ_1 and λ_2 using the constraints [3].

Therefore, by recasting our original problem of obtaining the boundary points of $\mathcal{R}(P, P_R)$ to the power minimization problem with SINR constraints, we have shown that the optimal solution can be found in an efficient way. In Section V, we plot the achievable rate region of the optimal AF strategy and compare it with the lower bound obtained by using dual channel matching, and an upper bound.

Recall that we only considered a two-way relay channel, where each relay had a single antenna, $N = 1$. Extension to $N > 1$, is straightforward by replacing $g_k w_k h_k$ by $\mathbf{g}_k \mathbf{W}_k \mathbf{h}_k$, $g_k w_k$ by $\mathbf{g}_k \mathbf{W}_k$, $h_k^r w_k g_k$ by $\mathbf{h}_k^r \mathbf{W}_k \mathbf{g}_k$ and $h_k^r w_k$ by $\mathbf{h}_k^r \mathbf{W}_k$, which are scalars as before, and the optimal solution to $\mathbf{W}_k$'s can be found using the iterative power minimization algorithm (6).

Our algorithm to optimize the achievable region with AF is fairly simple, however, it assumes that each relay has CSI for all the relay nodes, and requires $M = 1$. Finding optimal relay beamformers where each relay has only its CSI, and $M > 1$, is rather complicated and has not been solved even for the one-way relay channel [16]. Another limitation of the optimal AF strategy is that the expression for the obtained rate region cannot be written down in close form, and therefore does not allow analytical tractability for comparison with an upper bound. To remove these restrictions, in the next section we propose a simple AF strategy, called dual channel matching, where each relay uses its own CSI, and for which the achievable rate region expression can be written down in a closed form. Since dual channel matching is in general, a suboptimal AF strategy, the achievable rate region of dual channel matching lower bounds the rate region of the optimal AF strategy, and allows to estimate the difference between the optimal AF strategy and the upper bound.

IV. Dual Channel Matching Strategy

In this section we propose a simple AF strategy, called dual channel matching, and derive a lower bound on the achievable rate region for the two-way relay channel. With the dual channel matching strategy relay k multiplies $\sqrt{\beta_k}\left(\mathbf{G}_k^* \mathbf{H}_k^* + \mathbf{H}_k^{r*} \mathbf{G}_k^{r*}\right)$ to the received signal and forwards it to T_1 and T_2, where β_k is the normalization constant to satisfy the power constraint. Dual channel matching tries to match both the channels which the data streams from T_1 to T_2 and T_2 to T_1 experience at each relay node. The motivation for this strategy is that for one-way relay channel (i.e. T_2 has no data for T_1) with one relay node, the optimal AF strategy is to multiply $\mathbf{V}_2 \mathbf{D} \mathbf{U}_1^*$ to the signal at the relay, where the singular value decomposition of $\mathbf{H}_1$ is $\mathbf{U}_1 \mathbf{D}_1 \mathbf{V}_1^*$ and $\mathbf{G}_1$ is $\mathbf{U}_2 \mathbf{D}_2 \mathbf{V}_2^*$ and $\mathbf{D}$ is a diagonal matrix whose entries are chosen by waterfilling [29]. In dual channel matching the complex conjugates of the channels

[3]Clearly, the optimal $\mathbf{w}$ lies in the null space of some matrix that is a function of $\mathbf{A}, \mathbf{B}, \mathbf{c}, \mathbf{e}$, and $\mathbf{D}$ and hence not unique.

are used directly rather than the unitary matrices from the SVD of the channels [29]. This modification makes it easier to analyze the achievable rates for the two-way relay channel. Note that the dual channel matching is an extension of the listen and transmit strategy of [30] for the one-way relay channel, where each relay transmits the received signal after scaling it with the complex conjugates of the forward and backward channel coefficients.

Together with dual channel matching we restrict the signal transmitted from T_1 and T_2, $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, to be circularly symmetric complex Gaussian distributed with covariance matrix $\mathbb{E}\{\mathbf{x}_1\mathbf{x}_1{}^*\} = \mathbb{E}\{\mathbf{x}_2\mathbf{x}_2{}^*\} = \mathbf{I}_M$, to obtain a lower bound on the achievable rate region of two-way relay channel. Moreover, we use $\alpha = \frac{1}{2}$ i.e. T_1 and T_2 transmit and receive for same amount of time. The achievable rates R_{12} and R_{21} using the dual channel matching can be computed as follows.

From (1), the received signal at the k^{th} relay is given by

$$\mathbf{r}_k = \sqrt{\frac{P}{M}}\mathbf{H}_k\mathbf{x}_1 + \sqrt{\frac{P}{M}}\mathbf{G}_k^r\mathbf{x}_2 + \mathbf{n}_k.$$

Using dual channel matching as described above, at relay k, $\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}$ is multiplied to the received signal so that the transmitted signal $\mathbf{t}_k$ is given by

$$\mathbf{t}_k = \sqrt{\beta_k}\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{r}_k$$

where β_k is to ensure that $\sum_{k=1}^{K}\mathbf{t}_k^*\mathbf{t}_k = P_R$ [4]. With dual channel matching the received signal at T_2 is given by

$$\mathbf{y}_2 = \sum_{k=1}^{K}\mathbf{G}_k\mathbf{t}_k + \mathbf{z}. \tag{9}$$

Expanding (9) we can write

$$\begin{aligned}\mathbf{y} \;=\; & \underbrace{\sum_{k=1}^{K}\sqrt{\frac{P\beta_k}{M}}\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{H}_k}_{\mathbf{A}}\mathbf{x}_1 + \sum_{k=1}^{K}\sqrt{\frac{P\beta_k}{M}}\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{G}_k^r\mathbf{x}_2 \\ & + \sum_{k=1}^{K}\underbrace{\sqrt{\beta_k}\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)}_{\mathbf{B}_k}\mathbf{n}_k + \mathbf{z}.\end{aligned}$$

Since $\mathbf{x}_2$ and all the channel coefficients are known at T_2, the second term can be removed from the received signal at T_2. Moreover, as described before $\mathbf{x}_1$ is circularly symmetric complex Gaussian vector with covariance matrix $\mathbf{Q} = \mathbf{I}_M$, thus the achievable rate for T_1 to T_2 link is [31]

$$R_{12} = \frac{1}{2}I(\mathbf{x}_1;\mathbf{y}_2) = \frac{1}{2}\log\det\left(\mathbf{I}_M + \mathbf{A}\mathbf{A}^*\left(\sum_{k=1}^{K}\mathbf{B}_k\mathbf{B}_k^* + \mathbf{I}_M\right)^{-1}\right), \tag{10}$$

[4]This is for the sum power constraint. For an individual power constraint, β is chosen such that $\mathbf{t}_k^*\mathbf{t}_k = P_R$ for each k.

since $\mathbb{E}\left\{\mathbf{n}_k\mathbf{n}_k^*\right\} = \mathbb{E}\left\{\mathbf{z}\mathbf{z}^*\right\} = \mathbf{I}_M, \ \forall \ k$. Similarly, we obtain the expression for R_{21},

$$R_{21} = \frac{1}{2}\log\det\left(\mathbf{I}_M + \mathbf{C}\mathbf{C}^*\left(\sum_{k=1}^{K}\mathbf{D}_k\mathbf{D}_k^* + \mathbf{I}_M\right)^{-1}\right), \tag{11}$$

where $\mathbf{C} = \sum_{k=1}^{K}\sqrt{\frac{P\beta_k}{M}}\mathbf{H}_k^r\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{G}_k^r$ and $\mathbf{D}_k = \sqrt{\beta_k}\mathbf{H}_k^r\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)$. This rate region expression obtained is analytically tractable and can be used to compare the loss between the optimal AF strategy and the upper bound. Another interesting question of interest is how does the achievable rate region behaves with K. To answer that question, we turn to asymptotics and compute the rate region in the limit $K \to \infty$, in the next lemma.

Lemma 1: As K grows large, $K \to \infty$,

$$\lim_{K\to\infty} R_{12} \underset{w.p.1}{=} \frac{M}{2}\log K + \mathcal{O}(1),$$
$$\lim_{K\to\infty} R_{21} \underset{w.p.1}{=} \frac{M}{2}\log K + \mathcal{O}(1).$$

Proof: Consider

$$\begin{aligned} 2R_{12} - \log\det K\mathbf{I}_M &= \log\det\left(\mathbf{I}_M + \mathbf{A}\mathbf{A}^*\left(\sum_{k=1}^{K}\mathbf{B}_k\mathbf{B}_k^* + \mathbf{I}_M\right)^{-1}\right) - \log\det K\mathbf{I}_M, \\ &= \log\det\left(\frac{1}{K}\mathbf{I}_M + \frac{\mathbf{A}}{\sqrt{K}}\frac{\mathbf{A}^*}{\sqrt{K}}\left(\sum_{k=1}^{K}\mathbf{B}_k\mathbf{B}_k^* + \mathbf{I}_M\right)^{-1}\right). \end{aligned}$$

To satisfy the sum power constraint, let $\beta = \frac{P_R}{c_1 K}$ [5], where c_1 is a constant such that

$$c_1 = \mathbb{E}\left\{\left(\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{r}_k\right)^*\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{r}_k\right\},$$

which is same for all k. Then,

$$\frac{\mathbf{A}}{\sqrt{K}} = \sqrt{\frac{PP_R}{c_1 M}}\frac{1}{K}\sum_{k=1}^{K}\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\mathbf{H}_k,$$

which by using strong law of large numbers, converges to,

$$\frac{\mathbf{A}}{\sqrt{K}} \xrightarrow{w.p.1} \sqrt{\frac{PP_R}{c_1 M}}N^2\mathbf{I}_M,$$

since $\mathbb{E}\left\{\mathbf{G}_k\mathbf{G}_k^*\right\} = \mathbb{E}\left\{\mathbf{H}_k^*\mathbf{H}_k\right\} = N\mathbf{I}_M \ \forall \ k$, and $\mathbb{E}\left\{\mathbf{G}_k\mathbf{H}_k^{r*}\right\} = 0\mathbf{I}_M \ \forall \ k$. Same result holds true for $\frac{\mathbf{A}^*}{\sqrt{K}}$. With $\beta = \frac{P_R}{c_1 K}$,

$$\sum_{k=1}^{K}\mathbf{B}_k\mathbf{B}_k^* = \frac{P_R}{c_1}\frac{1}{K}\sum_{k=1}^{K}\left(\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\right)\left(\mathbf{G}_k\left(\mathbf{G}_k^*\mathbf{H}_k^* + \mathbf{H}_k^{r*}\mathbf{G}_k^{r*}\right)\right)^*,$$

[5] Equal power allocation among relays.

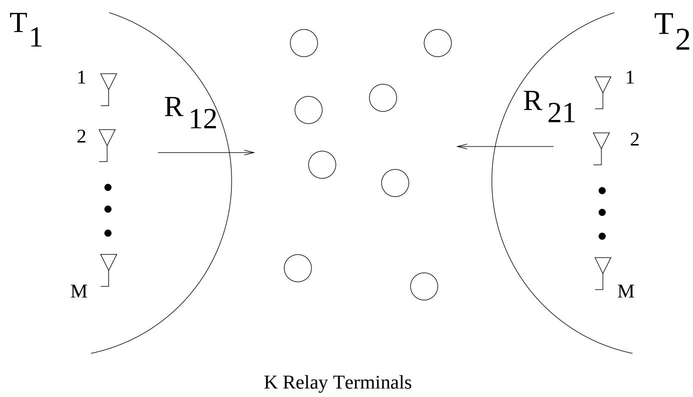

Fig. 5. Broadcast Cut

which again using the strong law of large numbers converges to $\frac{P_R}{c_1}\theta \mathbf{I}_M$, for some finite θ, since $\mathbf{H}_k, \mathbf{G}_k, \mathbf{H}_k^r, \mathbf{G}_k^r \mathbf{B}_k^*$ are i.i.d. with finite variance. Thus, in the limit $K \to \infty$,

$$2R_{12} - \log\det K\mathbf{I}_M \to M \log\left(\frac{PP_R N^4 c_1}{M(P_R\theta + c_1)}\right),$$

and thus it follows that

$$R_{12} \underset{w.p.1}{=} \frac{M}{2}\log K + \mathcal{O}(1). \tag{12}$$

Similarly we get the achievable rate R_{21} on the T_2 to T_1 link as

$$\lim_{K\to\infty} R_{21} \underset{w.p.1}{=} \frac{M}{2}\log K + \mathcal{O}(1). \tag{13}$$

■

Discussion: In this section we introduced the dual channel matching AF strategy, and obtained a lower bound on the capacity region of the two-way relay channel. Dual channel matching is a simple AF strategy that requires local CSI, and as we will see in Section V, has achievable rate region very close to that of the optimal AF strategy (Section III) for $M = 1$. We also derived the asymptotic achievable rate region of the dual channel matching, by taking the limit $K \to \infty$, and using the law of large numbers. We showed, that in the asymptotic regime, both R_{12} and R_{21} scale as $\frac{M}{2}\log K$ with increasing K.

Next, we derive an upper bound on the capacity region of the two-way relay channel, and compare it with the achievable rate region of the dual channel matching.

V. Upper Bound on the Two-Way Relay Channel Capacity

In this section we upper bound the capacity region of the two-way relay channel using the cut-set bound [19] for the broadcast cut, and the multiple access cut. We assume a general two-phase protocol where for α fraction of the time slot T_1 and T_2 transmit to all relays and the rest of the $(1-\alpha)$ fraction of time slot all relays simultaneously transmit to both T_1 and T_2. Note that to lower bound the capacity of the two-way relay channel using dual channel matching, we used $\alpha = \frac{1}{2}$ which might be suboptimal. We prove later that for the asymptotic case of $K \to \infty$, $\alpha = \frac{1}{2}$ is optimal.

The upper bound is derived as follows. We start by first separating T_1 and then T_2 from the network and apply the cut set bound [19] to upper bound the rate of information transfer between $T_1 \to T_2$ and $T_2 \to T_1$, respectively. Using the cutset bound, we first show that the maximum rate at of information transfer from $T_1 \to T_2$ ($T_2 \to T_1$) is upper bounded by the maximum rate of information transfer between T_1 (T_2) and $r_1, r_2, \ldots, r_K$ (broadcast cut) and also by the maximum rate of information transfer between $r_1, r_2, \ldots, r_K$ and T_2 (T_1) (multiple access cut), Fig. 5 and 6. Then we use the capacity results from [31] to upper bound the maximum rate through the broadcast cut for the case when CSI is only available at the receiver (all relays) and all the relays collaborate to decode the information. Similarly, for the multiple access cut as shown in Fig. 6, we upper bound the maximum rate at which all the $r_1, r_2, \ldots r_K$ can communicate to T_2 (T_1) by using capacity results from [31], when CSI is known both at the transmitter (all relays) and the receiver (T_1, T_2) and all the relays collaborate to transmit the information.

Broadcast cut - To derive an upper bound we make use of the cutset bound (Section 14.10 [19]). Separating the terminal T_1 from the rest of the network and applying the cutset bound on the broadcast cut as shown in Fig. 5,

$$R_{12} \leq \alpha \left\{ I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K, \mathbf{y}_2 | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) \right\}. \tag{14}$$

Again applying the cutset bound while separating the terminal T_2,

$$R_{21} \leq \alpha \left\{ I(\mathbf{x}_2; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K, \mathbf{y}_1 | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_1) \right\} \tag{15}$$

for some joint distribution $p(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K, \mathbf{x}_2)$, where R_{12} and R_{21} are the maximum rates at which T_1 can communicate to T_2 and T_2 can communicate to T_1 respectively, reliably. By the definition of mutual information [19]

$$\begin{aligned} I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K, \mathbf{y}_2 | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) &= I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) \\ &\quad + I(\mathbf{x}_1; \mathbf{y}_2 | \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K,\ \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2). \end{aligned} \tag{16}$$

By expanding the mutual information in terms of entropy,

$$\begin{aligned} I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) &= h(\mathbf{x}_1 | \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) \\ &\quad - h(\mathbf{x}_1 | \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K, \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_K, \mathbf{x}_2) \end{aligned}$$

Since conditioning can only reduce entropy [19],

$$I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2) \leq h(\mathbf{x}_1|\mathbf{x}_2) - h(\mathbf{x}_1|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K,\mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2).$$

Note that $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_K$ is a function of $\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K$, which implies

$$I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2) \leq h(\mathbf{x}_1|\mathbf{x}_2) - h(\mathbf{x}_1|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K,\mathbf{x}_2)$$

and hence[6]

$$I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2) \leq I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2). \tag{17}$$

Given perfect channel knowledge at terminal T_2,

$$I(\mathbf{x}_1;\mathbf{y}_2|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K,\ \mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2) = I(\mathbf{x}_1,\mathbf{z}_2)$$

where $\mathbf{z}_2$ is the AWGN noise. Since $\mathbf{x}_1$ and $\mathbf{z}_2$ are independent, $I(\mathbf{x}_1,\mathbf{z}_2)=0$, and therefore from (16, 17),

$$I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K,\mathbf{y}_2|\mathbf{t}_1,\mathbf{t}_2,\ldots\mathbf{t}_K,\mathbf{x}_2) \leq I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2).$$

Hence from (14),

$$R_{12} \leq I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2). \tag{18}$$

Similarly, by interchanging the roles of $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$R_{21} \leq I(\mathbf{x}_2;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_1). \tag{19}$$

Therefore it is clear that both R_{12} and R_{21} is upper bounded by the maximum information flow through the broadcast cut Fig. 5 when all the relays are allowed to collaborate. Expanding the mutual information in terms of differential entropy,

$$I(\mathbf{x}_1;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2) = h(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2) - h(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_1,\mathbf{x}_2).$$

From (1),

$$\mathbf{r}_k = \sqrt{\frac{P}{M}}\mathbf{H}_k\mathbf{x}_1 + \sqrt{\frac{P}{M}}\mathbf{G}_k^r\mathbf{x}_2 + \mathbf{n}_k.$$

Since $\mathbf{G}_k^r$ is known at relay k,

$$h\left(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_K|\mathbf{x}_2\right) = h\left(\sqrt{\frac{P}{M}}\mathbf{H}_1\mathbf{x}_1+\mathbf{n}_1,\sqrt{\frac{P}{M}}\mathbf{H}_2\mathbf{x}_1+\mathbf{n}_2,\ldots,\sqrt{\frac{P}{M}}\mathbf{H}_K\mathbf{x}_1+\mathbf{n}_K|\mathbf{x}_2\right).$$

[6]Without $\mathbf{x}_2$, in [20], this inequality has been shown to be an equality, which is incorrect.

Since conditioning can only decrease entropy,

$$h\left(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_2\right) \leq h\left(\sqrt{\frac{P}{M}}\mathbf{H}_1\mathbf{x}_1 + \mathbf{n}_1, \sqrt{\frac{P}{M}}\mathbf{H}_2\mathbf{x}_1 + \mathbf{n}_2, \ldots, \sqrt{\frac{P}{M}}\mathbf{H}_K\mathbf{x}_1 + \mathbf{n}_K\right).$$

With perfect knowledge of $\mathbf{H}_k$ and $\mathbf{G}_k^r$ at relay k,

$$h\left(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_1, \mathbf{x}_2\right) = h\left(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_K\right),$$

and it follows that

$$\begin{aligned} I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_2) \quad \leq \quad & h\left(\sqrt{\frac{P}{M}}\mathbf{H}_1\mathbf{x}_1 + \mathbf{n}_1, \sqrt{\frac{P}{M}}\mathbf{H}_2\mathbf{x}_1 + \mathbf{n}_2, \ldots, \sqrt{\frac{P}{M}}\mathbf{H}_K\mathbf{x}_1 + \mathbf{n}_K\right) \\ & -h\left(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_K\right). \end{aligned} \tag{20}$$

Thus, we have shown that R_{12} is upper bounded by the maximum rate from T_1 to $r_1, \ldots, r_K$ without any interference from T_2 and when all r_k's can collaborate to decode the message, which is quite intuitive. Using results from [31] when CSI is known only at the receiver, the R.H.S. of (20) is upper bounded by $\log\det\left(\mathbf{I}_M + \sum_{k=1}^{K} \frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right)$, which implies

$$I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_2) \leq \alpha \log\det\left(\mathbf{I}_M + \sum_{k=1}^{K} \frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right) \tag{21}$$

and therefore, from (18)

$$R_{12} \leq \alpha I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_2) \leq \alpha \log\det\left(\mathbf{I}_M + \sum_{k=1}^{K} \frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right). \tag{22}$$

Interchanging the roles of $\mathbf{x}_1$ and $\mathbf{x}_2$ and replacing $\mathbf{H}_k$ with $\mathbf{G}_k$,

$$R_{21} \leq \alpha I(\mathbf{x}_2; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K | \mathbf{x}_1) \leq \alpha \log\det\left(\mathbf{I}_M + \sum_{k=1}^{K} \frac{P}{KM}\mathbf{G}_k^{r*}\mathbf{G}_k^r\right). \tag{23}$$

Multiple access cut - Again by using the cutset bound, we bound the maximum rate of information transfer R_{12} (R_{21}) from $T_1 \rightarrow T_2$ ($T_1 \rightarrow T_2$) by the maximum rate of information transfer across the multiple access cut as shown in Fig. 6. Using cutset bound, R_{12} and R_{21} are bounded by

$$R_{12} \leq (1-\alpha) I(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K; \mathbf{y}_2 | \mathbf{x}_2) \tag{24}$$

$$R_{21} \leq (1-\alpha) I(\mathbf{x}_2, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K; \mathbf{y}_1 | \mathbf{x}_1). \tag{25}$$

Now,

$$\begin{aligned} I(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K; \mathbf{y}_2 | \mathbf{x}_2) \quad = \quad & h(\mathbf{y}_2|\mathbf{x}_2) - h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K, \mathbf{x}_2) \\ & + h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K, \mathbf{x}_2) - h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K, \mathbf{x}_1, \mathbf{x}_2). \end{aligned}$$

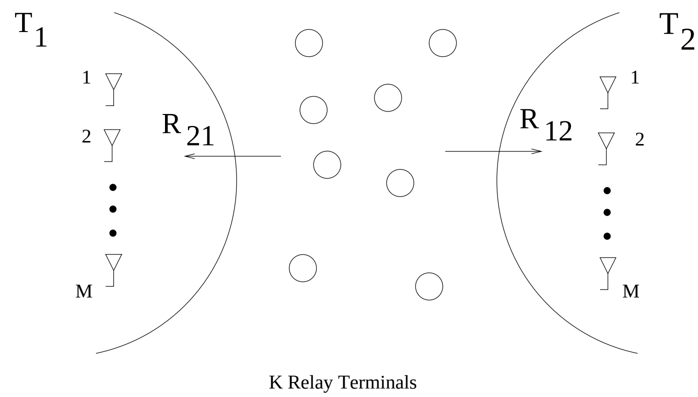

Fig. 6. Multiple Access Cut

Note that given $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K$, $\mathbf{y}_2$ is independent of $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K, \mathbf{x}_1, \mathbf{x}_2) = h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K, \mathbf{x}_2) = h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K).$$

Therefore

$$I(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_2|\mathbf{x}_2) = h(\mathbf{y}_2|\mathbf{x}_2) - h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K).$$

Since conditioning can only reduce entropy,

$$I(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_2|\mathbf{x}_2) \leq h(\mathbf{y}_2) - h(\mathbf{y}_2|\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K),$$

and by definition of mutual information

$$I(\mathbf{x}_1, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_2|\mathbf{x}_2) \leq I(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K, \mathbf{y}_2).$$

Hence from (24),

$$R_{12} \leq (1-\alpha) I(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_2). \tag{26}$$

Following similar steps we can also bound R_{21} as,

$$R_{21} \leq (1-\alpha) I(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_1). \tag{27}$$

Thus, R_{12}, R_{21} are bounded by the maximum rate of information from $r_1, \dots, r_K$ to T_1 or T_2. Next, we compute the maximum rate of information from $r_1, \dots, r_K$ to T_1 or T_2. Recall from (3) that the received signal $\mathbf{y}_2$ is given by

$$\mathbf{y}_2 = \sum_{k=1}^{K} \mathbf{G}_k \mathbf{t}_k + \mathbf{z}_2.$$

Note that

$$I(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \mathbf{y}_2) = I\left(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \frac{\mathbf{y}_2}{\sqrt{K}}\right).$$

Dividing $\mathbf{y}_2$ by $\sqrt{K}$, we get

$$\frac{\mathbf{y}_2}{\sqrt{K}} = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \mathbf{G}_k \mathbf{t}_k + \frac{\mathbf{z}_2}{\sqrt{K}}.$$

This can also be written as

$$\frac{\mathbf{y}_2}{\sqrt{K}} = \underbrace{\frac{1}{\sqrt{K}} [\mathbf{G}_1 \ \mathbf{G}_2 \ \ldots \ \mathbf{G}_K]}_{\Phi} [\mathbf{t}_1 \mathbf{t}_2 \ldots \mathbf{t}_K]^T + \frac{\mathbf{z}_2}{\sqrt{K}}.$$

Note that Φ is a $M \times NK$ matrix. Now assuming that all the relays know $\mathbf{G}_k, \ \forall k$ (allowing cooperation among all relays), with sum power available across all relays bounded by P_R, we have from [31],

$$R_{12} \leq (1-\alpha) I \left(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_K; \frac{\mathbf{y}_2}{\sqrt{K}} \right) \leq (1-\alpha) \sum_{l=1}^{\min\{NK,M\}} \max\{0, \log(K\lambda_l \nu)\} \tag{28}$$

where $\lambda_l, l = 1, 2, \ldots, \min\{NK, M\}$ are the eigen values of $\Phi\Phi^*$ matrix and ν is chosen such that

$$\sum_{l=1}^{\min\{NK,M\}} \max\{0, \nu - \frac{1}{\lambda_l}\} = P_R.$$

Similarly, one can obtain the bound for R_{21} by replacing $\mathbf{G}_k$ by $\mathbf{H}_k^r$.

Combining (22), (23) and (28), gives the upper bound on the capacity region of the two-way relay channel. Comparing the upper bound with the lower bound obtained using the dual channel matching (10,11), one can see that they do not match for any arbitrary value of K. In the asymptotic regime, however, they can be shown to be only an $\mathcal{O}(1)$ term away as $K \to \infty$, as proved in the next Theorem. This asymptotic result implies two things, one that the performance of the dual channel matching, and consequently the optimal AF strategy (which we don't know for $M > 1$), does not degrade in comparison to the upper bound with increasing K, and two, it provides us with the capacity scaling law of the two-way relay channel.

In Figs. 7 and 8, we plot the achievable rate region of the optimal AF strategy, the lower bound obtained using dual channel matching, and the upper bound for $K = 2$ and $K = 4$, with $M = 1, \ N = 1$ and $P = P_R = 10dB$ with sum rate constraint across relays. Note that the achievable rate region of the optimal AF region is symmetric, as expected, because of the symmetry in parameters of communication in both directions in a two-way relay channel. Another important point to note here is that, the achievable rates of dual channel matching are quite close to that of the optimal AF strategy, even though it uses only local CSI. Thus, dual channel matching is a good candidate for AF in practical implementation of two-way relay channels. Also notice that the difference between the upper and lower bound is less than the 3 bit bound of [9].

Next, we prove that the lower bound (dual channel matching) and the upper bound on the achievable rate region of the two-way relay channel are only an $\mathcal{O}(1)$ as $K \to \infty$. We prove the theorem by

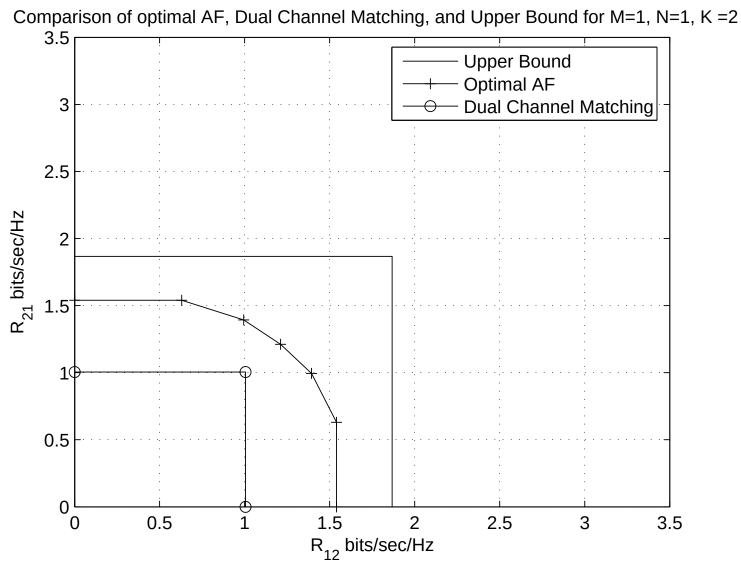

Fig. 7. Comparison of upper and lower bound of the capacity region of the two-way relay channel with $K = 2, M = 1, N = 1,$ $P = P_R = 10dB$

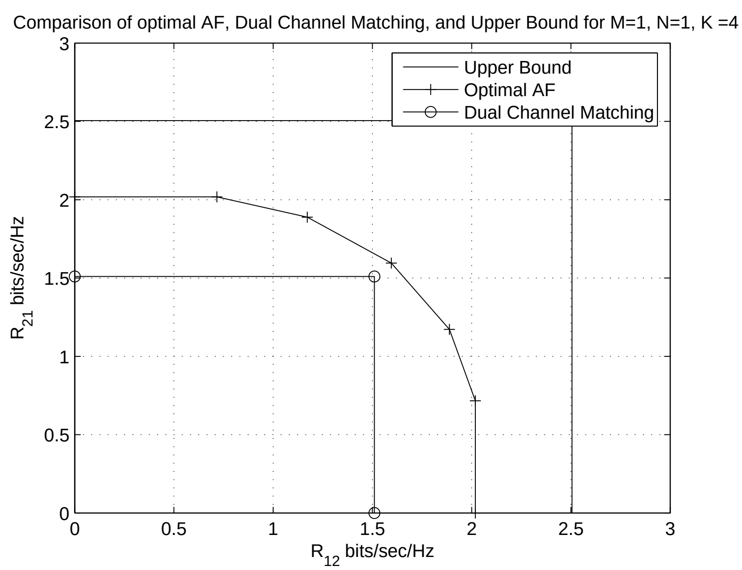

Fig. 8. Comparison of upper and lower bound of the capacity region of the two-way relay channel with $K = 4, M = 1, N = 1,$ $P = P_R = 10dB$

approximating the upper bound in the $K \to \infty$ and comparing it with the asymptotic lower bound obtained in (12, 13).

Theorem 1: The upper and lower bounds on the capacity region of the two-way relay channel differ by a $\mathcal{O}(1)$ term as $K \to \infty$, and the capacity scaling law is given by

$$\begin{aligned} R_{12} &\leq \frac{M}{2}\log K + \mathcal{O}(1), \\ R_{12} &\leq \frac{M}{2}\log K + \mathcal{O}(1). \end{aligned}$$

Proof: We first approximate the broadcast cut upper bound (22) as $K \to \infty$. From (22)

$$R_{12} \leq \alpha I(\mathbf{x}_1; \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_K | \mathbf{x}_2) \leq \alpha \log\det\left(\mathbf{I}_M + \sum_{k=1}^{K} \frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right). \quad (29)$$

Consider

$$\log\det\left(\mathbf{I}_M + \sum_{k=1}^{K}\frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right) - \log\det K\mathbf{I}_M = \log\det\left(\frac{1}{K}\mathbf{I}_M + \frac{1}{K}\sum_{k=1}^{K}\frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right).$$

Using strong law of large numbers

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k \xrightarrow{w.p.1} \frac{PN}{M}\mathbf{I}_M, \text{ since } \mathbb{E}\{\mathbf{H}_k^*\mathbf{H}_k\} = N\mathbf{I}_M,$$

and it follows that

$$\log\det\left(\frac{1}{K}\mathbf{I}_M + \frac{1}{K}\sum_{k=1}^{K}\frac{P}{M}\mathbf{H}_k^*\mathbf{H}_k\right) \to M\log\left(\frac{PN}{M}\right),$$

which using (22) implies

$$\lim_{K\to\infty} R_{12} \underset{w.p.1}{\leq} \alpha M\log K + \mathcal{O}(1), \quad (30)$$

since M, N, P are finite integers. Similarly,

$$\lim_{K\to\infty} R_{21} \underset{w.p.1}{\leq} \alpha M\log K + \mathcal{O}(1). \quad (31)$$

Next, we approximate the upper bound of the multiple access cut. From (28),

$$R_{12} \leq (1-\alpha) I\left(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_K; \frac{\mathbf{y}_2}{\sqrt{K}}\right) \leq (1-\alpha)\sum_{l=1}^{\min\{NK,M\}} \max\{0, \log(K\lambda_l\nu)\}, \quad (32)$$

where $\lambda_l, l = 1, 2, \dots, \min\{NK, M\}$ are the eigen values of $\Phi\Phi^*$ matrix and ν is chosen such that

$$\sum_{l=1}^{\min\{NK,M\}} \max\{0, \nu - \frac{1}{\lambda_l}\} = P_R.$$

By definition $\Phi\Phi^* = \frac{1}{K}\sum_{k=1}^{K}\mathbf{G}_k\mathbf{G}_k^*$. From strong law of large numbers

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\mathbf{G}_k\mathbf{G}_k^* \xrightarrow{w.p.1} N\mathbf{I}_M.$$

Therefore

$$\lambda_i = N \ \ \forall\ i = 1, 2, \ldots M, \implies \nu = \left(\frac{P_R}{M} + \frac{1}{N}\right),$$

and from (32)

$$R_{12} \underset{w.p.1}{\leq} (1-\alpha) \sum_{l=1}^{M} \log\left(KN\left(\frac{P_R}{M} + \frac{1}{N}\right)\right),$$

and consequently, as $K \to \infty$

$$R_{12} \underset{w.p.1}{\leq} (1-\alpha) M \log K + \mathcal{O}(1), \tag{33}$$

and similarly

$$R_{21} \underset{w.p.1}{\leq} (1-\alpha) M \log K + \mathcal{O}(1). \tag{34}$$

Combining (30,31) and (33,34)

$$\begin{aligned} R_{12} &\leq \min\{\alpha, 1-\alpha\} M \log K + \mathcal{O}(1) \leq \frac{M}{2} \log K + \mathcal{O}(1), \\ R_{21} &\leq \min\{\alpha, 1-\alpha\} M \log K + \mathcal{O}(1) \leq \frac{M}{2} \log K + \mathcal{O}(1). \end{aligned} \tag{35}$$

Comparing (35) to the asymptotic lower bound (12, 13) we conclude that (a) upper and lower bounds on the capacity region of the two-way relay channel differ by a $\mathcal{O}(1)$ term as $K \to \infty$, and (b) the capacity scaling law is given by

$$\begin{aligned} R_{12} &\leq \frac{M}{2} \log K + \mathcal{O}(1), \\ R_{21} &\leq \frac{M}{2} \log K + \mathcal{O}(1). \end{aligned}$$

■

To illustrate the result of Theorem 1, in Fig.9, we compare the lower (dual channel matching) and upper bound on the sum rate $R_{12} + R_{21}$, and show that they both scale similarly with increasing K for $M = 2, N = 1$, $P = P_R = 10dB$ with sum rate constraint across relays.

Discussion: In this section we obtained upper bounds on the capacity region of the two-way relay channel, and compared it with the dual channel matching lower bound. To compute the upper bound we used the cut-set bound and the capacity results of [31]. The lower and upper bound expressions do not match in general, however, in the asymptotic case, where the number of relays are large, $K \to \infty$, we showed that they are only an $\mathcal{O}(1)$ term away from each other. Thus, the dual channel matching and consequently, the optimal AF strategy are almost optimal in the asymptotic regime. For the finite number of relay nodes (finite K), we use Monte Carlo simulations to quantify the gap between the lower and the upper bound. From Figs. 7 and 8, we can see that gap between the lower (dual channel matching) and upper bound is rather small, and inside the 3 bit bound of [9].

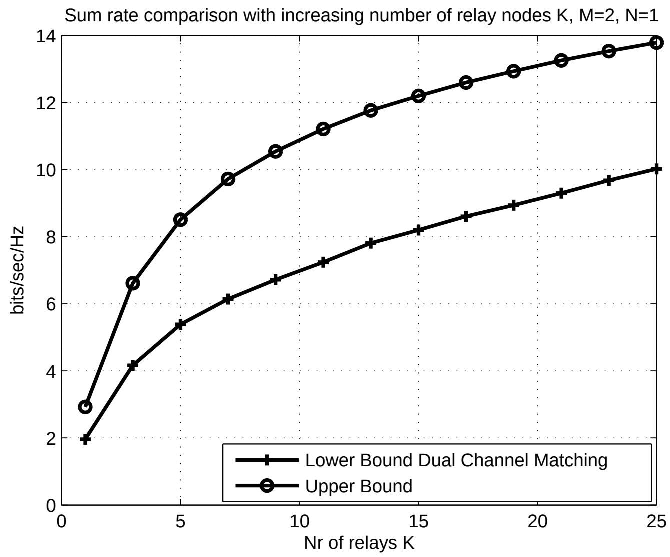

Fig. 9. Capacity scaling of two-way relay channel with $M = 2, N = 1, P = P_R = 10dB$.

Another important observation to make is that the lower bound with dual channel matching was obtained using $\alpha = \frac{1}{2}$ i.e. T_1 and T_2 transmit and receive for equal amount of time. Since this lower bound is only a $\mathcal{O}(1)$ term away from the upper bound (arbitrary α), distributing equal amount of time for transmit and receive phase is optimal in achieving the right capacity scaling.

Compared to the asymptotic results on the one-way relay channel [20], [32], our results show that by two-way relay channel one can remove the $\frac{1}{2}$ rate loss factor on the capacity, which comes from the half-duplex assumption on the terminals and relays. Therefore with two-way relay channel one can achieve unidirectional full-duplex performance with half-duplex terminals.

VI. DIVERSITY-MULTIPLEXING TRADEOFF

In this section we consider a two-way relay channel with a single relay node, and characterize its DM-tradeoff. We consider both the full-duplex and half-duplex nodes, where T_1 and T_2 have m_1 and m_2 antennas, respectively, and the relay node has m_r antennas. An important difference in this section from the previous ones is the presence of direct link between T_1 and T_2 as shown in Fig. 4.

To characterize the DM-tradeoff, for both the full-duplex and half-duplex case, we first obtain an upper bound on the DM-tradeoff and then propose a modified CF strategy to achieve the upper bound. We first discuss the full-duplex case followed by the half-duplex case.

A. DM-tradeoff of Full-Duplex Two-Way Relay Channel

The signal model for this section is as follows. Let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_r$ be the signal transmitted from T_1, T_2 and the relay node, respectively. Similarly, Let $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_r$ be the signal received at T_1, T_2 and the relay node, respectively. Recall that channel coefficient between T_1 and relay node is denoted by $\mathbf{H}$, between T_1 and T_2 is denoted by $\mathbf{H}_{12}$, between the relay node and T_2 is denoted by $\mathbf{G}$, where note that, compared to previous sections, we have dropped the subscript index of relay node, since we only consider one relay. All the channel coefficients in the reverse direction (right to left) are denoted by channel coefficient in the forward direction (left to right) with an added superscript r, e.g. the channel coefficients between the relay node and T_1 is denoted by $\mathbf{H}^r$. Let the transmit power at T_1, T_2 and the relay node be P[7]. Then,

$$\begin{aligned}
\mathbf{y}_1 &= \sqrt{\frac{P}{m_2}}\mathbf{H}_{12}^r\mathbf{x}_2 + \sqrt{\frac{P}{m_r}}\mathbf{H}^r\mathbf{x}_r + \mathbf{n}_1, \\
\mathbf{y}_2 &= \sqrt{\frac{P}{m_1}}\mathbf{H}_{12}\mathbf{x}_1 + \sqrt{\frac{P}{m_r}}\mathbf{G}\mathbf{x}_r + \mathbf{n}_2, \\
\mathbf{y}_r &= \sqrt{\frac{P}{m_1}}\mathbf{H}\mathbf{x}_1 + \sqrt{\frac{P}{m_2}}\mathbf{G}^r\mathbf{x}_2 + \mathbf{n}_r. \qquad (36)
\end{aligned}$$

Let the rate of transmission from T_1 to T_2 and T_2 to T_1 be R_{12} and R_{21}, respectively. Following [21], let $\mathcal{C}_{12}(\mathsf{SNR})$ and $\mathcal{C}_{21}(\mathsf{SNR})$ be the family of codes, one for each SNR for transmission from T_1 to T_2, and T_2 to T_1, respectively. Then we define r_{12} (r_{21} similarly) as the multiplexing gain of $\mathcal{C}_{12}(\mathsf{SNR})$ if the data rate $R_{12}(\mathsf{SNR})$ ($R_{21}(\mathsf{SNR})$) of $\mathcal{C}_{12}(\mathsf{SNR})$ ($\mathcal{C}_{21}(\mathsf{SNR})$) scales as r_{12} (r_{21}) with respect to $\log \mathsf{SNR}$, i.e.

$$\lim_{\mathsf{SNR}\to\infty} \frac{R_{12}(\mathsf{SNR})}{\log \mathsf{SNR}} = r_{12}$$

and $d_{12}(r_{12}, r_{21})$ ($d_{21}(r_{12}, r_{21})$) as the rate of fall of probability of error P_{e12} (P_{e21}) of $\mathcal{C}_{12}(\mathsf{SNR})$ ($\mathcal{C}_{21}(\mathsf{SNR})$) with respect to SNR, i.e.

$$P_{e12}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d_{12}(r_{12}, r_{21})}.$$

The negative of the SNR exponent of the error probability $d_{12}(r_{12}, r_{21})$ or $d_{21}(r_{12}, r_{21})$ captures the DM-tradeoff, where $d_{12}(r_{12}, r_{21})$ ($d_{21}(r_{12}, r_{21})$) is the maximum diversity gain possible from T_1 to T_2 (T_2 to T_1) for a given r_{12} and r_{21}. Note that the error probability $P_{e12}(\mathsf{SNR})$ and $P_{e21}(\mathsf{SNR})$ are functions of both r_{12} and r_{21} because of simultaneous transmission between T_1 and T_2.

Next, we upper bound the DM-tradeoff of the two-way relay channel, the region spanned by $d_{12}(r_{12}, r_{21})$ and $d_{21}(r_{12}, r_{21})$, by allowing cooperation between T_1 and relay, and T_2 and relay node.

[7]Having different transmit power constraints for T_1, T_2 and the relay node do not change the DM-tradeoff.

Lemma 2: The DM-tradeoff of a two-way relay channel is upper bounded by

$$d_{12}(r_{12}, r_{21}) \leq \min\{(m_1 - r_{12})(m_r + m_2 - r_{12}),\ (m_1 + m_r - r_{12})(m_2 - r_{12})\},$$

$$d_{21}(r_{12}, r_{21}) \leq \min\{(m_2 - r_{21})(m_r + m_1 - r_{21}),\ (m_2 + m_r - r_{21})(m_1 - r_{21})\},\ \forall r_{12},\ r_{21}.$$

Proof: We will prove the lemma only for $d_{12}(r_{12}, r_{21})$. For $d_{21}(r_{12}, r_{21})$ it follows similarly. Consider the case when T_2 has no data to send to T_1. This assumption can only improve $d_{12}(r_{12}, r_{21})$. Then first assume that the relay node and T_2 are co-located and can cooperate perfectly. In this case, the communication model from T_1 to T_2 is a point to point MIMO channel with m_1 transmit antennas and $m_r + m_2$ receive antennas. The DM-tradeoff of this MIMO channel is $(m_1 - r_{12})(m_r + m_2 - r_{12})$, and since this point to point MIMO channel is better than our original two-way relay channel, $d_{12}(r_{12}, r_{21}) \leq (m_1 - r_{12})(m_r + m_2 - r_{12})$ [8] . Next, we assume that T_1 is co-located with relay node and both of them can perfectly cooperate for transmission to T_2. This setting is equivalent to a MIMO channel with $m_1 + m_r$ transmit and m_2 receive antenna with DM-tradeoff $(m_1 + m_r - r_{12})(m_2 - r_{12})$. Again, this point to point MIMO channel is better than our original two-way relay channel and hence $d_{12}(r_{12}, r_{21}) \leq (m_1 + m_r - r_{12})(m_2 - r_{12})$, which completes the proof. ■

To achieve this upper bound we consider the CF strategy [27], with a slight modification and prove that it is sufficient, to achieve the optimal DM-tradeoff. We make few changes to the original CF strategy [27] to suit the two-way relay channel communication, which are as follows. Let the rate of transmission from T_1 to T_2 and T_2 to T_1 be R_{12} and R_{21}, respectively. Instead of generating only one codebook at T_1 as in [27], both T_1 and T_2 generate $2^{nR_{12}}$ and $2^{nR_{21}}$ independent and identically distributed x_1^n and x_2^n according to distribution $p(x_1^n) = \prod_{i=1}^n p(x_{1i})$ and $p(x_2^n) = \prod_{i=1}^n p(x_{2i})$, respectively. The codebook generation at the relay and the relay compression and transmission remains the same as in [27], i.e. the relay node generates 2^{nR_0} independent and identically distributed x_r^n according to distribution $p(x_r^n) = \prod_{i=1}^n p(x_{ri})$ and label them $x_r(s),\ s \in [1, 2^{nR_0}]$, and for each $x_r(s)$ generates $2^{n\hat{R}}$ $\hat{y}$'s, each with probability $p(\hat{y}|x_r(s)) = \prod_{i=1}^n p(\hat{y}_i|x_{ri}(s))$. Label these $\hat{y}(z|s), s \in [1, 2^{nR_0}]$ and $z \in [1, 2^{n\hat{R}}]$ and randomly partition the set $[1, 2^{n\hat{R}}]$ into 2^{nR_0} cells $S_s,\ s \in [1, 2^{nR_0}]$. Let in block i the message to send from T_1 is w_i, and from T_2 is v_i, then T_1 sends $x_1(w_i)$, T_2 sends $x_2(v_i)$ and the relay sends $x_r(s_i)$ if $z_i \in s_i$, where $\hat{y}(z_i|s_{i-1}), y_r(i-1), x_r(s_{i-1})$ are jointly typical. Decoding at both T_1 and T_2 remains the same as in [27], however, note that in this case T_1 knows $x_1(w_i)$ and T_2 knows $x_2(v_i)$ apriori and

[8]This upper bound is valid as long as the coherence time T_c is smaller than the time it takes for T_2 to compute the channel coefficients and feed them back to T_1, which is at least $m_1 + m_2$ [33]. Otherwise, T_2 can help T_1 in acquiring transmit CSI, for which case, potentially infinite diversity gain can be achieved [34], violating the present upper bound.

therefore can use them to decode v_i and w_i respectively. This strategy has been previously considered in [8] to obtain achievable rate region.

With this two-way CF strategy, the following rates are achievable,

$$\begin{aligned} R_{12} &\leq I(\mathbf{x}_1; \mathbf{y}_2 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2), \\ R_{21} &\leq I(\mathbf{x}_2; \mathbf{y}_1 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_1), \end{aligned} \tag{37}$$

with the compression rate constraint

$$\max\{I(\mathbf{y}_r; \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_1 \mathbf{y}_1), I(\mathbf{y}_r; \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2 \mathbf{y}_2)\} \leq \min\{I(\mathbf{x}_r; \mathbf{y}_1 | \mathbf{x}_1), I(\mathbf{x}_r; \mathbf{y}_2 | \mathbf{x}_r \mathbf{x}_2)\}. \tag{38}$$

The rate region and the compression constraint are a little different from [27]. The rate region differs because of conditioning by $\mathbf{x}_1$ or $\mathbf{x}_2$, which is due to the prior knowledge of $\mathbf{x}_1$ at T_1 and $\mathbf{x}_2$ at T_2. The new compression rate constraint incorporates the condition that the quantized version of $\mathbf{y}_r$, $\hat{\mathbf{y}}_r$ can be decoded at both T_1 and T_2. In the next Theorem we compute the outage exponents for (37) and show that they match with the exponents of the upper bound.

Theorem 2: CF strategy achieves the DM-tradeoff upper bound (Lemma 2).

Proof: To prove the Theorem we will compute the achievable DM-tradeoff of the CF strategy (37) and show that it matches with the upper bound.

To compute the achievable rates subject to the compression rate constraints for the signal model (36), we fix $\hat{\mathbf{y}} = \mathbf{y}_r + \mathbf{n}_q$, where $\mathbf{n}_q$ is $m_r \times 1$ vector with covariance matrix $\hat{N}\mathbf{I}_{m_r}$. Also, we choose $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_r$ to be complex Gaussian with covariance matrices $\frac{P}{m_1}\mathbf{I}_{m_1}$, $\frac{P}{m_2}\mathbf{I}_{m_2}$, and $\frac{P}{m_r}\mathbf{I}_{m_r}$, and independent of each other. respectively. Next, we compute the various mutual information expressions to derive the achievable DM-tradeoff of the CF strategy. By the definition of the mutual information

$$I(\mathbf{x}_1; \mathbf{y}_2 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2) \quad = \quad h(\mathbf{y}_2 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2) - h(\mathbf{y}_2 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2 \mathbf{x}_1).$$

From (36), arranging $\mathbf{y}_2$ $\hat{\mathbf{y}}$ in a vectorized form we get

$$\left[\begin{array}{c} \mathbf{y}_2 \\ \hat{\mathbf{y}} \end{array}\right] = \left[\begin{array}{c} \sqrt{\frac{P}{m_1}}\mathbf{H}_{12}\mathbf{x}_1 + \sqrt{\frac{P}{m_r}}\mathbf{G}\mathbf{x}_r + \mathbf{n}_2, \\ \sqrt{\frac{P}{m_1}}\mathbf{H}\mathbf{x}_1 + \sqrt{\frac{P}{m_2}}\mathbf{G}^r\mathbf{x}_2 + \mathbf{n}_r + \mathbf{n}_q. \end{array}\right] \tag{39}$$

and consequently

$$h(\mathbf{y}_2 \hat{\mathbf{y}} | \mathbf{x}_r \mathbf{x}_2) = \log L_1^{r2}, \tag{40}$$

where

$$L_1^{r2} = \det\left(\frac{P}{m_1}\mathbf{H}_1^{r2}\mathbf{H}_1^{r2*} + \left[\begin{array}{cc} \left(\hat{N}+1\right)\mathbf{I}_{m_r} & 0 \\ 0 & \mathbf{I}_{m_2} \end{array}\right]\right) \text{ and } \mathbf{H}_1^{r2} = [\mathbf{H}_{12}\ \mathbf{H}].$$

Moreover, from (39)

$$h(\mathbf{y}_2\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{x}_1) = \log\det\left(\begin{bmatrix} (\hat{N}+1)\mathbf{I}_{m_r} & 0 \\ 0 & \mathbf{I}_{m_2} \end{bmatrix}\right),$$

which implies

$$I(\mathbf{x}_1;\mathbf{y}_2\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2) = \log\frac{L_1^{r2}}{(\hat{N}+1)^{m_r}}. \quad (41)$$

Similarly, one can show,

$$I(\mathbf{x}_2;\mathbf{y}_1\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_1) = \log\frac{L_2^{r1}}{(\hat{N}+1)^{m_r}},$$

where

$$L_2^{r1} = \det\left(\frac{P}{m_1}\mathbf{H}_2^{r1}\mathbf{H}_2^{r1*} + \begin{bmatrix} \left(\hat{N}+1\right)\mathbf{I}_{m_r} & 0 \\ 0 & \mathbf{I}_{m_2} \end{bmatrix}\right) \text{ and } \mathbf{H}_2^{r1} = [\mathbf{H}_{12}^r \ \mathbf{G}^r]. \quad (42)$$

Next, we compute the value of $\hat{N}$ that satisfies the compression rate constraints (38). By the definition of mutual information,

$$\begin{aligned} I(\mathbf{y}_r;\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2) &= h(\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2) - h(\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2\mathbf{y}_r) \\ &= h(\hat{\mathbf{y}}\mathbf{y}_2|\mathbf{x}_r\mathbf{x}_2) - h(\mathbf{y}_2|\mathbf{x}_r\mathbf{x}_2) - h(\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2\mathbf{y}_r). \quad (43) \end{aligned}$$

From (40), $h(\hat{\mathbf{y}}\mathbf{y}_2|\mathbf{x}_r\mathbf{x}_2) = \log L_1^{r2}$. From signal model (36), it is easy to see that $h(\mathbf{y}_2|\mathbf{x}_r\mathbf{x}_2) = \log L_{12}$, where $L_{12} = \det\left(\frac{P}{m_1}\mathbf{H}_{12}\mathbf{H}_{12}^* + I_{m_2}\right)$. Given $\mathbf{y}_r$, $\hat{\mathbf{y}}$ has only the noise term $\mathbf{n}_q$, and hence $h(\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_1\mathbf{y}_1\mathbf{y}_r) = \log\hat{N}^{m_r}$. Therefore, from (43),

$$I(\mathbf{y}_r;\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2) = \log\frac{L_1^{r2}}{L_{12}\hat{N}^{m_r}}. \quad (44)$$

Similarly one can compute

$$I(\mathbf{y}_r;\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_1\mathbf{y}_1) = \log\frac{L_2^{r1}}{L_{21}\hat{N}^{m_r}}, \text{ where } L_{21} = \det\left(\frac{P}{m_2}\mathbf{H}_{12}^r\mathbf{H}_{12}^{r*} + I_{m_1}\right). \quad (45)$$

Again using the definition of mutual information,

$$\begin{aligned} I(\mathbf{x}_r;\mathbf{y}_1|\mathbf{x}_1) &= h(\mathbf{y}_1|\mathbf{x}_1) - h(\mathbf{y}_1|\mathbf{x}_r\mathbf{x}_1) \\ &= \log L_{2r}^1 - \log L_{21}, \quad (46) \end{aligned}$$

where $L_{2r}^1 = \det\left(\frac{P}{m_2}\mathbf{H}_{12}^r\mathbf{H}_{12}^{r*} + \frac{P}{m_r}\mathbf{H}^r\mathbf{H}^{r*} + \mathbf{I}_{m_1}\right)$, since $\mathbf{y}_1 = \sqrt{\frac{P}{m_2}}\mathbf{H}_{12}^r\mathbf{x}_2 + \sqrt{\frac{P}{m_r}}\mathbf{H}^r\mathbf{x}_r + \mathbf{n}_1$. Similarly,

$$I(\mathbf{x}_r;\mathbf{y}_2|\mathbf{x}_2) = \log\frac{L_{1r}^2}{L_{12}}, \quad (47)$$

where $L_{1r}^2 = \det\left(\frac{P}{m_1}\mathbf{H}_{12}\mathbf{H}_{12}^* + \frac{P}{m_r}\mathbf{G}\mathbf{G}^* + \mathbf{I}_{m_2}\right)$.

To satisfy the compression rate constraints (38), from (44), (45), (46), (47), clearly

$$\hat{N} \geq \frac{\max\left\{\log \frac{L_1^{r2}}{L_{12}\hat{N}^{m_r}}, \log \frac{L_2^{r1}}{L_{21}\hat{N}^{m_r}}\right\}}{\min\left\{\log \frac{L_{1r}^2}{L_{12}}, \log \frac{L_{2r}^1}{L_{21}}\right\}}. \tag{48}$$

We choose $\hat{N}$ to satisfy the equality (48). From [21], to compute $d_{12}(r_{12}, r_{21})$, it is sufficient to find the negative of the exponent of the SNR of outage probability at T_2, where outage probability at T_2, $P_{out}(r_{12}\log \mathsf{SNR})$, is defined as

$$P_{out}(r_{12}\log \mathsf{SNR}) \quad = \quad P(R_{12} \leq r_{12}\log \mathsf{SNR})$$

From (37, 41),

$$R_{12} = \log \frac{L_1^{r2}}{(\hat{N}+1)^{m_r}}, \tag{49}$$

where $\hat{N}$ is given in (48). Then,

$$\begin{aligned} P_{out}(r_{12}\log \mathsf{SNR}) &= P\left(\log \frac{L_1^{r2}}{(\hat{N}+1)^{m_r}} \leq r_{12}\log \mathsf{SNR}\right), \\ &= P\left(\frac{L_1^{r2}}{(\hat{N}+1)^{m_r}} \leq \mathsf{SNR}^{r_{12}}\right). \end{aligned}$$

Choose $l \in \mathbb{Z}$ such that $(\hat{N}+1)^{m_r} \leq l\left(\left(\frac{L_1^{r2}}{L_{1r}^2}\right)^{1/m_r}+1\right)^{m_r}$, where $\hat{N}$ is such that it meets the equality in (48). Then,

$$\begin{aligned} P_{out}(r_{12}\log \mathsf{SNR}) &\dot{\leq} P\left(\frac{L_1^{r2}}{l\left(\left(\frac{L_1^{r2}}{L_{1r}^2}\right)^{1/m_r}+1\right)^{m_r}} \leq \mathsf{SNR}^{r_{12}}\right), && (50) \\ &= P\left(\left(\frac{(L_1^{r2})^{1/m_r}(L_{1r}^2)^{1/m_r}}{l^{1/m_r}\left((L_1^{r2})^{1/m_r}+(L_{1r}^2)^{1/m_r}\right)}\right)^{m_r} \leq \mathsf{SNR}^{r_{12}}\right), && (51) \\ &= P\left(\frac{(L_1^{r2})^{1/m_r}(L_{1r}^2)^{1/m_r}}{(L_1^{r2})^{1/m_r}+(L_{1r}^2)^{1/m_r}} \leq l^{1/m_r}\mathsf{SNR}^{r_{12}/m_r}\right), && (52) \\ &\dot{\leq} P\left(\frac{(L_1^{r2})^{1/m_r}(L_{1r}^2)^{1/m_r}}{(L_1^{r2})^{1/m_r}+(L_{1r}^2)^{1/m_r}} \leq \mathsf{SNR}^{r_{12}/m_r}\right), && (53) \end{aligned}$$

where the last equality follows because multiplying SNR by a constant does not change DM-tradeoff. From here on we follow [25] to compute the exponent of the $P_{out}(r_{12}\log \mathsf{SNR})$. Let

$$L_{1l}^{r2} = \det\left(\frac{P}{m_1}\mathbf{H}_1^{r2}\mathbf{H}_1^{r2*} + \mathbf{I}_{m_r+m_2}\right). \tag{54}$$

Then clearly from (40), $L_{1l}^{r2} \leq L_1^{r2}$, therefore using Lemma 2 [25], it follows that

$$P_{out}(r_{12}\log \mathsf{SNR}) \quad \leq \quad P\left(\frac{(L_{1l}^{r2})^{1/m_r}(L_{1r}^2)^{1/m_r}}{(L_{1l}^{r2})^{1/m_r}+(L_{1r}^2)^{1/m_r}} \leq \mathsf{SNR}^{r_{12}/m_r}\right). \tag{55}$$

Moreover, notice that for non-negative random variables X and Y and a constant c [25], $P(XY/(X+Y)<c) \leq P(X<2c)+P(Y<2c)$, thus,

$$P_{out}(r_{12}\log\mathsf{SNR}) \;\leq\; P\left((L_{1l}^{r2})^{1/m_r} \leq 2\mathsf{SNR}^{r_{12}/m_r}\right) + P\left((L_{1r}^{2})^{1/m_r} \leq 2\mathsf{SNR}^{r_{12}/m_r}\right), \quad (56)$$

$$\doteq\; P\left(L_{1l}^{r2} \leq \mathsf{SNR}^{r_{12}}\right) + P\left(L_{1r}^{2} \leq \mathsf{SNR}^{r_{12}}\right), \quad (57)$$

$$\doteq\; \mathsf{SNR}^{-d_1(r_{12})} + \mathsf{SNR}^{-d_2(r_{12})},$$

$$\doteq\; \mathsf{SNR}^{-\min\{d_1(r_{12}),\ d_2(r_{12})\}}. \quad (58)$$

Therefore, to lower bound the DM-tradeoff we need to find out the outage exponents $d_1(r_{12})$ and $d_2(r_{12})$ of L_{1l}^{r2} and L_{1r}^{2}. Notice that, however, L_{1l}^{r2} is the mutual information between T_1 and T_2 by choosing the covariance matrix to be $\frac{P}{m_1}\mathbf{I}_{m_1}$[9], and allowing the relay and T_2 to cooperate perfectly. From [21], choice of $\frac{P}{m_1}\mathbf{I}_{m_1}$ as the covariance matrix does not change the optimal DM-tradeoff, therefore, $d_1(r_{12}) = (m_1-r_{12})(m_r+m_2-r_{12})$. Similar argument holds for L_{1r}^{2}, by noting that L_{1r}^{2} is the mutual information between T_1 and T_2 if the relay and T_1 were co-located and could cooperate perfectly, while using covariance matrix $\frac{P}{m_1+m_r}\mathbf{I}_{m_1+m_r}$. Thus, $d_2(r_{12}) = (m_1+m_r-r_{12})(m_2-r_{12})$. Thus, for T_1 to T_2 communication, the achievable DM-tradeoff with CF strategy meets the upper bound (Lemma 2). A similar result can be obtained for T_2 to T_1 communication by choosing an appropriate $n \in \mathbb{Z}$ such that $(\hat{N}+1)^{m_r} \leq n\left(\left(\frac{L_2^{r1}}{L_{2r}^{1}}\right)^{1/m_r}+1\right)^{m_r}$, where $\hat{N}$ is such that it meets the equality in (48) and by carrying out the outage exponent analysis of $R_{21} = \log\frac{L_2^{r1}}{(\hat{N}+1)^{m_r}}$ and lower bounding L_2^{r1} by L_{2l}^{r1}, where $L_{2l}^{r1} = \det\left(\frac{P}{m_2}\mathbf{H}_2^{r1}\mathbf{H}_2^{r1*}+\mathbf{I}_{m_r+m_1}\right)$. ■

B. Half-Duplex Two-Way Relay Channel

In this section we compute the DM-tradeoff of the half-duplex two-way relay channel where all the nodes (T_1, T_2 and the relay) are half- duplex. For the half-duplex case, the achievable rate regions are protocol dependent and the optimal protocol is unknown in general [3]–[5]. Here we compute the DM-tradeoff of a three phase protocol, that is intuitively optimal (difficult to prove), where for t_1 fraction of the time slot T_1 transmits to both T_2 and the relay, t_2 fraction of the time slot T_2 transmits to T_1 and the relay, and for the rest $(1-t_1+t_2)$ fraction of the time slot the relay transmits to both T_1 and T_2.

For this communication protocol the rates R_{12} and R_{21} are upper bounded by the following expressions.

$$R_{12} \leq \max_{t_1,t_2}\min\left\{t_1 I(\mathbf{x}_1;\mathbf{y}_r,\mathbf{y}_2), t_1 I(\mathbf{x}_1;\mathbf{y}_2)+(1-t_1-t_2)I(\mathbf{x}_r;\mathbf{y}_2)\right\},$$

$$R_{21} \leq \max_{t_1,t_2}\min\left\{t_2 I(\mathbf{x}_2;\mathbf{y}_r,\mathbf{y}_1), t_2 I(\mathbf{x}_2;\mathbf{y}_1)+(1-t_1-t_2)I(\mathbf{x}_r;\mathbf{y}_1)\right\},$$

[9] P taking the role of SNR.

where the first argument in the minimum is obtained by allowing the relay and the T_2 (T_1) to collaborate in the receive mode, and the second argument is obtained by simply adding the maximum mutual information possible at T_2 (T_1) while in receiving mode. Using the rate region expression, we define the upper bound on the DM-tradeoff of the half-duplex two-way relay channel as follows.

From the definition of L_{1l}^{r2} (54),

$$\begin{aligned} P\left(t_1 I(\mathbf{x}_1;\mathbf{y}_r,\mathbf{y}_2) \le r_{12}\log \mathsf{SNR}\right) &\doteq P\left(t_1 \log L_1^{r2} \le r_{12}\log \mathsf{SNR}\right), \\ &:= \mathsf{SNR}^{-d_{bc}^{12}(r_{12})}, \text{ and} \end{aligned} \tag{59}$$

$$\begin{aligned} P\left(t_1 I(\mathbf{x}_1;\mathbf{y}_2) + (1-t_1-t_2) I(\mathbf{x}_r;\mathbf{y}_2)\right) &\doteq P\left(t_1 \log L_{12} + (1-t_1-t_2)\log L_{r2} \le r_{12}\log \mathsf{SNR}\right), \\ &:= \mathsf{SNR}^{-d_{mac}^{12}(r_{12})}, \end{aligned} \tag{60}$$

where $L_{2r} = \det\left(\mathbf{I}_{m_2} + \frac{P}{m_r}\mathbf{G}\mathbf{G}^*\right)$. Thus, $d_{12}(r_{12}, r_{21}) \le \max_{t_1,t_2} \min\left\{d_{bc}^{12}(r_{12}),\ d_{mac}^{12}(r_{12})\right\}$. Similarly we can obtain upper bound for $d_{21}(r_{12}, r_{21})$ by replacing t_1 by t_2 in (59, 60).

To achieve this upper bound we consider the CF strategy of subsection VI-A, except that in this case the compression signal $\hat{y}$ is chosen such that it is jointly typical with the received signals y_{rt_1} and y_{rt_2} received in time t_1 and t_2 from T_1 and T_2, respectively [10]. With this CF strategy the achievable rate region is given by

$$\begin{aligned} R_{12} &\le t_1 I(\mathbf{x}_1; \mathbf{y}_2\hat{\mathbf{y}}|\mathbf{x}_r, \mathbf{x}_2)), \\ R_{21} &\le t_2 I(\mathbf{x}_2; \mathbf{y}_1\hat{\mathbf{y}}|\mathbf{x}_r, \mathbf{x}_1)), \end{aligned}$$

subject to the following compression rate constraint

$$(t_1+t_2)\max\{I(\mathbf{y}_r;\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_1\mathbf{y}_1), I(\mathbf{y}_r;\hat{\mathbf{y}}|\mathbf{x}_r\mathbf{x}_2\mathbf{y}_2)\} \le (1-(t_1+t_2))\min\{I(\mathbf{x}_r;\mathbf{y}_1|\mathbf{x}_1), I(\mathbf{x}_r;\mathbf{y}_2|\mathbf{x}_r\mathbf{x}_2)\}. \tag{61}$$

To compute these rates, we let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_r$ to be the same as in the full-duplex case and $\hat{\mathbf{y}} = \mathbf{y}_{rt_1} + \mathbf{y}_{rt_2} + \mathbf{n}_q$, where $\mathbf{n}_q$ is the complex Gaussian vector with zero mean and covariance matrix $\hat{N}\mathbf{I}_r$. Following the same steps as in (48) to (57), we obtain

$$\begin{aligned} P(R_{12} \le r_{12}\log\mathsf{SNR}) &\dot{\le} P(t_1 \log L_{1l}^{r2} \le r_{12}\log\mathsf{SNR}) + \\ &\quad P\left(\frac{(2(t_1+t_2)-1)t_1}{t_1+t_2}\log L_{12} + \frac{(1-(t_1+t_2))t_1}{t_1+t_2}\log L_{1r}^2 \le r_{12}\log\mathsf{SNR}\right), \\ &:= \mathsf{SNR}^{-d_{bc}^{12}(r_{12})} + \mathsf{SNR}^{-d_{mac}^{12'}(r_{12})}, \\ &\doteq \mathsf{SNR}^{-\min\left\{d_{bc}^{12}(r_{12}), d_{mac}^{12'}(r_{12})\right\}}. \end{aligned} \tag{62}$$

[10] In [4] a similar strategy has been proposed, but there, two separate compression signals are chosen that are jointly typical with y_{rt_1} and y_{rt_2} individually, and then a deterministic function of the two compression signals is transmitted from the relay, which results in a different rate region expression from the one obtained here.

Thus the achievable $d_{12}(r_{12}, r_{21}) \leq \max_{t_1,t_2} \min\left\{d_{bc}^{12}(r_{12}), d_{mac}^{12'}(r_{12})\right\}$. Note that the expression for $d_{12}(r_{12}, r_{21})$ is independent of r_{21}, and because of symmetry in R_{12} and R_{21} expressions, similar bounds can be obtained for R_{21} by replacing t_1 with t_2, and is given by

$$\begin{aligned} P(R_{21} \leq r_{21} \log \mathsf{SNR}) &\leq P(t_1 \log L_{2l}^{r1} \leq r_{21} \log \mathsf{SNR}) + \\ & \quad P\left(\frac{(2(t_1+t_2)-1)t_2}{t_1+t_2} \log L_{12} + \frac{(1-(t_1+t_2))t_2}{t_1+t_2} \log L_{2r}^{1} \leq r_{21} \log \mathsf{SNR}\right), \\ &:= \mathsf{SNR}^{-d_{bc}^{21}(r_{21})} + \mathsf{SNR}^{-d_{mac}^{21'}(r_{21})}, \end{aligned}$$

which implies

$$d_{21}(r_{12}, r_{21}) \leq \max_{t_1,t_2} \min\left\{d_{bc}^{21}(r_{21}), d_{mac}^{21'}(r_{21})\right\}. \tag{63}$$

It is clear that the lower bound (62, 63) and the upper bound (59,60) on the DMT of the half-duplex two-way relay channel do not match for the general case. By comparing the achievable DM-tradeoff and the upper bound, the next Theorem characterizes the cases for which CF strategy is optimal.

Theorem 3: The proposed CF strategy achieves the optimal DM-tradeoff of the half-duplex two way relay channel if

- the bottleneck of the channel is the broadcast cut, i.e. $d_{bc}^{12}(r_{12}) \leq d_{mac}^{12'}(r_{12})$ and correspondingly in the upper bound $d_{bc}^{12}(r_{12}) \leq d_{mac}^{12}(r_{12})$, and with similar relation for $d_{bc}^{21}(r_{21})$ and $d_{mac}^{21}(r_{21})$also.
- otherwise if $\frac{(2(t_1+t_2)-1)t_1}{t_1+t_2} \log L_{12} + \frac{(1-(t_1+t_2))t_1}{t_1+t_2} \log L_{1r}^{2} = t_1 \log L_{12} + (1-t_1-t_2)L_{r2}$, and with similar relation for T_2 to T_1 communication.

Proof: Follows immediately by comparing the lower bound (62) and the upper bound (59,60) on the DM-tradeoff. ∎

Discussion: In this section we showed that the CF strategy achieves the optimal DM-tradeoff of the two-way relay channel for the full-duplex case, in general, and for the half-duplex case in some cases. For both the full-duplex and half-duplex case we upper bounded the DM-tradeoff allowing different nodes to collaborate with each other while transmitting or receiving. For the full-duplex case, we modified the CF strategy of [27] [11] and showed that it decouples the two-way relay channel into two one-way relay channel and achieves optimal DM-tradeoff on each of the two one-way relay channels. For the half-duplex case, as observed before, the achievable rate region and consequently the DM-tradeoff depends on the communication protocol. We used a three phase protocol that makes use of all the direct links between T_1, T_2, and the relay. For the three phase protocol we proposed a modified CF strategy where the compression signal is chosen such that it is jointly typical with the signals received at the relay node

[11] The same strategy can also be found in [4]

in phase 1 and 2. Using this CF strategy, we obtained a lower bound on the DM-tradeoff that is shown to match with the upper bound under some conditions. For the general case also, we believe that the proposed CF should be optimal in terms of achieving the DM-tradeoff, however, showing that is quite difficult because of the different mutual information quantities involved as well as the maximization over the time durations of phase 1 and 2.

Our result for the full-duplex case is similar to [25], where it is shown that the CF strategy achieves the optimal DM-tradeoff in one-way relay channel. For the half-duplex case, however, because of three phase communication protocol and added compression rate constraints we are unable to reach the same conclusion of [25] in general, that CF achieves the optimal DM-tradeoff in half-duplex one-way relay channel.

VII. Conclusion

In the first part of the paper, we addressed the problem of finding optimal relay beamformers to maximize the achievable rate region of the two-way relay channel with multiple relays, when each relay uses AF. The use of AF strategy is motivated by the fact that all the other known relay strategies such as DF, partial DF and CF, do not work well in the presence of multiple relays, and moreover, AF is quite simple to implement.

For the case when both the terminals T_1 and T_2 have a single antenna and each relay has an arbitrary number of antennas, we found an iterative algorithm to compute the optimal relay beamformers. The algorithm is equivalent to solving a power minimization problem subject to SINR constraints at each step. The power minimization problem at each step is non-convex, however, for which it is sufficient to satisfy the KKT conditions to obtain the optimal solution.

The derived optimal AF strategy maximizes the rate region with AF, but is restricted to the case of a single antenna at T_1 and T_2, and cannot be extended easily for the multi-antenna case. Moreover, it also requires each relay to have global CSI, and does not have a closed form achievable rate region expression. To relax the single antenna restriction and global CSI requirement, we then proposed a dual channel matching strategy, which requires local CSI, and showed that the gap between the rate region of the optimal AF and dual channel matching is quite small when both T_1 and T_2 have a single antenna. The dual channel matching works for any number of antennas at T_1 and T_2, and has a closed form expression for the achievable rate region. We then compared the achievable rate region of the dual channel matching with an upper bound to quantify the loss while using dual channel matching. The analytical expressions of the lower and the upper bound did not match, and we used simulations to show that the gap is quite small. In the asymptotic regime of $K \to \infty$, however, using the analytical expressions, we proved that the achievable rate region of the dual channel matching, is only a constant term away from the upper bound. Thus, we obtained the capacity scaling law for the two-way relay channel. Compared to [20], [32], our capacity scaling law for the two-way relay channel shows that with two-way relay channel, there is a two-fold increase in the capacity compared to unidirectional communication.

In the second part of the paper, we considered the problem of finding coding strategies that achieve the optimal DM-tradeoff in a two-way relay channel with a single relay node, in the presence of direct path between T_1 and T_2. We showed that the CF strategy achieves the optimal DM-tradeoff of the full-duplex two-way relay channel, by first decoupling the two-way relay channel into two one-way relay channels, and achieving the optimal DM-tradeoff on each of the two one-way relay channel. For the half-duplex case we showed that a modified CF strategy for a three phase transmission protocol achieves the optimal DM-tradeoff for some cases.

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