# Capacity Scaling for MIMO Two-Way Relaying

Rahul Vaze and Robert W. Heath Jr.
The University of Texas at Austin
Department of Electrical and Computer Engineering
Wireless Networking and Communications Group
1 University Station C0803
Austin, TX 78712-0240
email: vaze@ece.utexas.edu, rheath@ece.utexas.edu

***Abstract*— This paper considers capacity scaling in the recently proposed two-way MIMO (multiple input multiple output) relay channel. In the two-way relay channel, two nodes use a relay for exchanging data with each other. Under the assumption that each node has perfect channel state information and all nodes work only in half duplex mode, this paper shows that the capacity scales linear with the number of transmit antennas and logarithmically with the number of relays, as the number of relays grows large. This result shows that with two-way relay channels it is possible to asymptotically (in the number of relays) obtain full-duplex performance while using only half-duplex nodes.**

## I. INTRODUCTION

Relay channels are a basic component of multi-hop networks where a relay node assists the transmission from a source to a destination [1], [2]. Several different relaying protocols have been proposed with different objectives including increasing capacity [4], [5], [6], [7] or enhancing coverage through cooperative diversity [8], [9]. Recently, the MIMO (multiple input multiple output) relay channel has attracted a lot of interest [10], [11], where the source, relay, and destination have multiple antennas. Upper and lower bound on the capacity of MIMO relay channel are derived in [3]. Although the capacity of MIMO relay channel with fixed number of relays is unknown, capacity scaling behavior of MIMO relay channel has been established in [14], when the number of relays grow large.

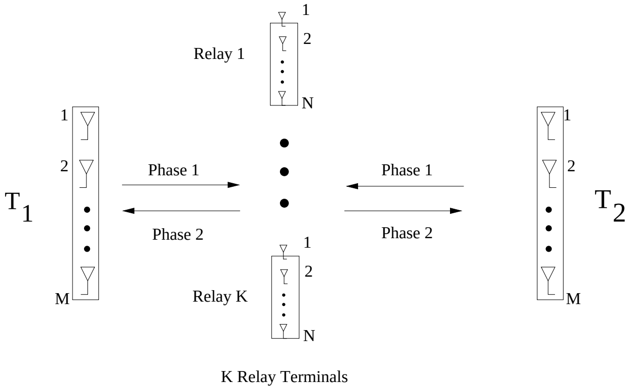

Fig. 1. Two way relaying protocol with multiple antennas.

An enhancement of the MIMO relay channel is the MIMO two-way relay channel, introduced in [15]. In the MIMO two-way channel, two nodes use relays to exchange messages with each other, so each node acts both as a source and a destination. A system block diagram for a MIMO two-way relay is provided in Fig. 1, with two nodes $T_1$ and $T_2$ who want to exchange information. In the first time slot, both $T_1$ and $T_2$ are scheduled to transmit simultaneously while the relays receive. In the next time slot, relays are scheduled to transmit and both the terminals $T_1$ and $T_2$ receive. The terminals use knowledge of the signals they sent in the first time-slot to cancel the known interference they receive from the relay in the second time slot. For this two-way relaying protocol, with using a single relay $K = 1$ and assuming that all the nodes are full-duplex (receive and transmit at the same time), achievable rate regions for $T_1$ and $T_2$ are derived in [16] for different protocols (amplify and forward, decode and forward, compress and forward). These achievable rates do not match with the best known upper bound [17] and the capacity region of two-way relaying is unknown in general. Specializing this result for the case of Gaussian channels and half-duplex nodes, [15] shows that by using two-way relaying one can remove the $\frac{1}{2}$ rate loss factor in spectral efficiency due to half duplex assumption on the nodes. Even for this special case, the capacity region is unknown.

The goal of this paper is to study the capacity scaling behavior of large networks [12], [13], [14], with two-way relaying for Gaussian channels. We consider the same two-way protocol as proposed by [15] with $K$ relays, where no relay has any data of its own. We assume that both $T_1$ and $T_2$ are equipped with $M$ antennas, while all the $K$ relays have $N$ antennas each. We also assume that perfect channel state information (CSI) is available at all the nodes, Under these assumptions, the contributions of this paper can be summarized as follows:

- We derive an upper bound on the two-way relaying capacity and show that it scales as $M\log(K) + \mathcal{O}(1)$ for any $N$, as $K \to \infty$, for some joint encoding between $T_1$ and $T_2$.
- We derive a lower bound on the two-way relaying capacity and show that it is possible to achieve the upper bound on the coherent two-way relaying capacity upto a $\mathcal{O}(1)$ term, without any cooperation between $T_1$ and $T_2$.

Combining our upper and lower bounds, our main contribution is a characterization of the MIMO two-way relaying capacity as the number of relays grow large.

*Notation:* Throughout this paper the following notation is used. The superscripts ${}^{T}, {}^{H}$ represent the transpose and transpose conjugate. $\mathbf{M}$ denotes a matrix, $\mathbf{m}$ a vector and $m_i$ the $i^{th}$ element of $\mathbf{m}$. $\mathcal{E}(f)$ denotes the expectation of function $f$, $||.||$ denotes the usual Euclidean norm of a vector. $\mathbf{I}_m$ is a $m \times m$ identity matrix. We use the usual notation for $u(x) = \mathcal{O}(v(x))$ if $|\frac{u(x)}{v(x)}|$ remains bounded, as $x \to \infty$. $x \sim \mathcal{CN}(0, \sigma)$ means $x$ is a circularly symmetric complex Gaussian random variable with zero mean and variance $\sigma$. Variance of a random variable $a$ is denoted by var$(a)$. $\mathcal{C}^{M \times N}$ denotes the set of $M \times N$ matrices with complex entries. $x \xrightarrow{w.p.1} y$ denotes that random variable $x$ converges to $y$ with probability 1. $I(x; y)$ denotes the mutual information between $x$ and $y$, $H(x)$ the entropy of $x$ and $h(x)$ the differential entropy of $x$. For the definition of $I(x; y), H(x)$ and $h(x)$ we refer the reader to [20].

## II. TWO-WAY RELAYING SYSTEM MODEL

### A. Protocol

We consider a wireless network where there are two terminals $T_1$ and $T_2$ who want to communicate to each other via $K$ relays. The $K$ relays do not have any data of their own but only help $T_1$ and $T_2$ to exchange information. We assume that all the $K$ relays are located randomly and uniformly in a area of fixed size. We assume that both the terminals $T_1$ and $T_2$ have $M$ antennas and all the $K$ relays have $N$ antennas each. $N \geq 1$ and is independent of $M$. We assume that there exists no direct path between $T_1$ and $T_2$ and that they can communicate only through the $K$ relays. The transmission from terminals $T_1$ and $T_2$ is in spatial multiplexing mode i.e. $M$ antennas at each terminal, transmit $M$ independent data streams. We further assume that both the terminals and all the relays can operate only in half-duplex mode (cannot transmit and receive at the same time). The two-way relaying protocol is summarized as follows [15]. In the first time slot, both $T_1$ and $T_2$ are scheduled to transmit and all the relays receive a superposition of the signals transmitted from $T_1$ and $T_2$. In the next time slot, all the relays are scheduled to transmit simultaneously and both the terminals receive.

### B. Channel and Signal Model

We represent the forward channel between $T_1$ and the $k^{th}$ relay by $\mathbf{H}_k = [\mathbf{h}_{1k}\ \mathbf{h}_{2k}\ \ldots\ \mathbf{h}_{Mk}]$ and the backward channel between $k^{th}$ relay and $T_1$ by $\mathbf{H}_k^{(r)} = [\mathbf{h}_{k1}^{(r)}\ \mathbf{h}_{k2}^{(r)}\ \ldots\ \mathbf{h}_{kM}^{(r)}]$. Similarly the forward channel between $k^{th}$ relay and $T_2$ is denoted by $\mathbf{G}_k = [\mathbf{g}_{k1}\ \mathbf{g}_{k2}\ \ldots\ \mathbf{g}_{kM}]$ and the backward channel between $T_2$ and the $k^{th}$ relay is denoted by $\mathbf{G}_k^{(r)} = [\mathbf{g}_{1k}^{(r)}\ \mathbf{g}_{2k}^{(r)}\ \ldots\ \mathbf{g}_{Mk}^{(r)}]$. Therefore $\{\mathbf{H}_k, \mathbf{G}_k^{(r)}\} \in \mathbb{C}^{N \times M}$ and $\{\mathbf{H}_k^{(r)}, \mathbf{G}_k\} \in \mathbb{C}^{M \times N}$ with each entry in the matrix consisting of i.i.d $\mathcal{CN}(0, 1)$. We further assume that all these channels are frequency flat, slow fading channels and independently varying across integer multiples of time slots.

Under these assumptions, the received signal at $k^{th}$ relay in the first time slot, is given by

$$\mathbf{r}_k = \sqrt{\frac{E_k}{M}}\mathbf{H}_k\mathbf{x} + \sqrt{\frac{F_k}{M}}\mathbf{G}_k^{(r)}\mathbf{u} + \mathbf{n}_k \tag{1}$$

where $\mathbf{r}_k$ is the $N \times 1$ received signal vector, while $E_k$ and $F_k$ are the average signal energy received at the $k^{th}$ relay from $T_1$ and $T_2$ respectively. The signals $\mathbf{x}$ and $\mathbf{u}$ are transmitted from $T_1$ and $T_2$ which are to decoded at $T_2$ and $T_1$ respectively. We assume that the signal vectors $\mathbf{x}$ and $\mathbf{u}$ are $M \times 1$ circularly symmetric complex Gaussian with $\mathcal{E}(\mathbf{x}^H\mathbf{x}) = M = \mathcal{E}(\mathbf{u}^H\mathbf{u})$. The noise $\mathbf{n}_k$ is the $N \times 1$ spatio-temporal white complex Gaussian noise independent across relays with $\mathcal{E}(\mathbf{n}_k\mathbf{n}_k^H) = \sigma^2\mathbf{I}_N$. Each relay processes its incoming signal to transmit a $N \times 1$ signal $\mathbf{t}_k$ in the second time slot. The $M \times 1$ received signal $\mathbf{v}$ and $\mathbf{y}$ at terminal $T_1$ and $T_2$ respectively, in the second time slot are given by

$$\mathbf{v} = \sum_{k=1}^{K}\sqrt{\frac{Q_k}{N}}\mathbf{H}_k^{(r)}\mathbf{t}_k + \mathbf{w}, \quad \mathbf{y} = \sum_{k=1}^{K}\sqrt{\frac{P_k}{N}}\mathbf{G}_k\mathbf{t}_k + \mathbf{z} \tag{2}$$

where $Q_k, P_k$ denote the average signal energy received at $T_1$ and $T_2$ in the second time slot while $\mathbf{w}$ are $\mathbf{z}$ are $M \times 1$ spatio-temporal white complex Gaussian noise vectors with $\mathcal{E}(\mathbf{w}\mathbf{w}^H) = \mathcal{E}(\mathbf{z}\mathbf{z}^H) = \sigma^2\mathbf{I}_M$. We assume that there is a average power constraint at each of the relays, given by $\mathcal{E}(\mathbf{t}_k^H\mathbf{t}_k) = N$. Throughout this paper we assume that both $T_1$ and $T_2$ perfectly know $\{\mathbf{H}_k, \mathbf{H}_k^{(r)}, \mathbf{G}_k, \mathbf{G}_k^{(r)}\}\ \forall\ k$ in the receiving mode but not in the transmit mode. while each relay knows the CSI i.e. the $k^{th}$ relay knows either $\{\mathbf{H}_k, \mathbf{G}_k\}$ or $\{\mathbf{H}_k^{(r)}, \mathbf{G}_k^{(r)}\}$, in both the receive and the transmit mode. We call this setup as *coherent two-way relaying*.

The path loss and shadowing effect parameters $E_k, P_k, F_k$ and $Q_k\ \forall\ k$ are assumed to be independent random variables, strictly positive bounded and remain constant over the entire time period of interest. The randomness comes from the fact that the relays locations are chosen randomly and strict positivity comes from the fact that the communication is happening over a fixed area and bounded assumption comes from the fact that none of the relays are too close to either $T_1$ or $T_2$. This is a valid assumption, since it is a well known fact that in a random network with $K$ nodes uniformly distributed on a fixed two-dimensional area, the minimum distance between any two nodes in the network is larger than $\frac{1}{K^{1+\delta}}$ with high probability, for any $\delta > 0$ [18]. Following the same argument, $F_k$ and $Q_k$ $\forall\ k$ are also assumed to be independent, strictly positive and bounded random variables. The results in this paper apply to independent, positive and bounded $E_k, P_k, F_k, Q_k$.

## III. UPPER BOUND ON THE COHERENT TWO-WAY RELAYING CAPACITY

In this section we derive an upper bound on the capacity of coherent two-way relaying. [20].

*Theorem 1:* If the number of relays $K \to \infty$, the capacity of coherent two-way relaying is upper bounded by $M \log(K) + \mathcal{O}(1)$.

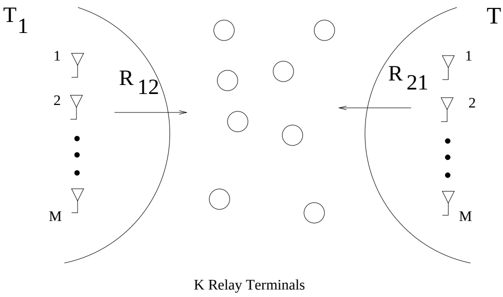

Fig. 2. Cut across both the sources

*Proof:* To prove the upper bound we make use of the cutset bound (Section 14.10 [20]). Separating the terminal $T_1$ first and then $T_2$, from the rest of the network and applying the cutset bounds on each cut, as shown in Fig. 2,

$$\begin{aligned} R_{12} \leq & \quad \mathcal{E}_{\{\mathbf{H}_k,\mathbf{G}_k\}_{k=1}^K} \left\{ \frac{1}{2} I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K,\mathbf{y}|\mathbf{t}_1,\ldots\mathbf{t}_K,\mathbf{u}) \right\} \\ R_{21} \leq & \quad \mathcal{E}_{\{\mathbf{H}_k^{(r)},\mathbf{G}_k^{(r)}\}_{k=1}^K} \left\{ \frac{1}{2} I(\mathbf{u};\mathbf{r}_1,\ldots,\mathbf{r}_K,\mathbf{v}|\mathbf{t}_1,\ldots\mathbf{t}_K,\mathbf{x}) \right\} \end{aligned} \tag{3}$$

for some joint distribution $p(\mathbf{x},\mathbf{t}_1,\ldots,\mathbf{t}_K,\mathbf{u})$, where $R_{12}$ and $R_{21}$ is the maximum rate at which $T_1$ can communicate to $T_2$ and $T_2$ can communicate to $T_1$ respectively, reliably. By chain rule of mutual Information [20]

$$\begin{aligned} & I(\mathbf{x};\mathbf{r_1},\ldots,\mathbf{r}_K,\mathbf{y}|\mathbf{t}_1,\ldots,\mathbf{t}_K,\mathbf{u}) = \\ & \quad I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{t}_1,\ldots,\mathbf{t}_K,\mathbf{u}) \\ & \quad + I(\mathbf{x};\mathbf{y}|\mathbf{r}_1,\ldots,\mathbf{r}_K,\ \mathbf{t}_1,\ldots,\mathbf{t}_K,\mathbf{u}). \end{aligned}$$

Since conditioning can only reduce entropy [20] and given $\{\mathbf{x},\mathbf{u}\}$, $\mathbf{r}_1,\ldots,\mathbf{r}_K$ are independent of $\mathbf{t}_1,\ldots,\mathbf{t}_K$, it follows that

$$I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{t}_1,\ldots,\mathbf{t}_K,\mathbf{u}) \leq I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{u}).$$

Given perfect channel knowledge at terminal $T_2$,

$$I(\mathbf{x};\mathbf{y}|\mathbf{r}_1,\ldots,\mathbf{r}_K,\ \mathbf{t}_1,\ldots\mathbf{t}_K,\mathbf{u}) = I(\mathbf{x},\mathbf{z})$$

where $\mathbf{z}$ is the AWGN noise. Since $\mathbf{x}$ and $\mathbf{z}$ are independent, $I(\mathbf{x},\mathbf{z}) = 0$, we have

$$R_{12} \leq \mathcal{E}_{\{\mathbf{H}_k,\mathbf{G}_k\}_{k=1}^K} \left\{ \frac{1}{2} I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{u}) \right\}.$$

Similarly we get the corresponding result for the $T_2 \rightarrow T_1$ link, by interchanging the roles of $\mathbf{x}$ and $\mathbf{u}$,

$$R_{21} \leq \mathcal{E}_{\{H_k^{(r)},\mathbf{G}_k^{(r)}\}_{k=1}^K} \left\{ \frac{1}{2} I(\mathbf{u};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{x}) \right\}.$$

Using the chain rule of mutual information and given $E_k, F_k, \mathbf{H}_k$ and $\mathbf{G}_k^{(r)}$ (which we assume is known at each of the relay), from (1), it follows that

$$\begin{aligned} & I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{u}) \leq \\ & \quad h\left(\sqrt{\frac{E_1}{M}}\mathbf{H}_1\mathbf{x}+\mathbf{n}_1,\ldots,\sqrt{\frac{E_K}{M}}\mathbf{H}_K\mathbf{x}+\mathbf{n}_K\right) \\ & \quad -h(\mathbf{n}_1,\ldots,\mathbf{n}_K). \end{aligned}$$

where the inequality follows from the fact that conditioning can only decrease entropy. Recalling that $\mathbf{x}$ was assumed to be circularly symmetric complex Gaussian with $\mathcal{E}(\mathbf{x}\mathbf{x}^H) = \mathbf{I}_M$, it follows from [19] that

$$I(\mathbf{x};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{u}) \leq \frac{1}{2}\log\det\left(\mathbf{I}_M + \frac{1}{\sigma^2}\sum_{k=1}^K \frac{E_k}{M}\mathbf{H}_k^H\mathbf{H}_k\right).$$

Interchanging the roles of $\mathbf{x}$ and $\mathbf{u}$ and replacing $E_k$ with $F_k$ and $\mathbf{H}_k$ with $\mathbf{G}_k$,

$$I(\mathbf{u};\mathbf{r}_1,\ldots,\mathbf{r}_K|\mathbf{x}) \leq \frac{1}{2}\log\det\left(\mathbf{I}_M + \frac{1}{\sigma^2}\sum_{k=1}^K \frac{F_k}{M}\mathbf{G}_k^{(r)H}\mathbf{G}_k^{(r)}\right).$$

Substituting these expressions in (3) and applying Jensen's inequality [20]

$$R_{12} \leq \frac{M}{2}\log\left(1+\frac{N}{M\sigma^2}\sum_{k=1}^K E_k\right) \triangleq R_{12}^U \tag{4}$$

and

$$R_{21} \leq \frac{M}{2}\log\left(1+\frac{N}{M\sigma^2}\sum_{k=1}^K F_k\right) \triangleq R_{21}^U. \tag{5}$$

Next we consider an approximation of this upper bound in the limit $K \rightarrow \infty$. Recall that both $E_k$ and $F_k$ are assumed to be bounded $\forall k$. This implies var$(E_k)$ and var$(F_k)$ are also bounded $\forall k$ and hence

$$\sum_{k=1}^{\infty} \frac{\text{var}(E_k)}{k^2} \leq \infty, \quad \sum_{k=1}^{\infty} \frac{\text{var}(F_k)}{k^2} \leq \infty.$$

Since the above sum is bounded for both $E_k$ and $F_k$, from (Theorem 1.8$D$ [21])

$$\sum_{k=1}^{\infty}\frac{E_k}{k} - \sum_{k=1}^{\infty}\frac{\mathcal{E}(E_k)}{k} \xrightarrow{w.p.1} 0, \quad \sum_{k=1}^{\infty}\frac{F_k}{k} - \sum_{k=1}^{\infty}\frac{\mathcal{E}(F_k)}{k} \xrightarrow{w.p.1} 0$$

Now since $\log(.)$ is a continuous function, using (Theorem 1.7 [21])

$$R_{12}^U \xrightarrow{w.p.1} \frac{M}{2}\log\left(\frac{KN\mathcal{E}(E_k)}{M\sigma^2}\right)$$

and

$$R_{21}^U \xrightarrow{w.p.1} \frac{M}{2}\log\left(\frac{KN\mathcal{E}(F_k)}{M\sigma^2}\right).$$

Given that $E_k, F_k$ are bounded $\forall k$ and $M, N$ are finite integers, as $K \rightarrow \infty$

$$C^U \triangleq R_{12}^U + R_{21}^U \xrightarrow{w.p.1} M\log(K) + \mathcal{O}(1)$$

■

## IV. LOWER BOUND ON COHERENT TWO-WAY RELAYING CAPACITY

In this section we lower bound the achievable rate for two-way relaying. We use a constructive protocol that involves maximal ratio combining (MRC) and beamforming for a single data stream at each relay. At each receiver ($T_1$ or $T_2$), every data stream is independently decoded by a single antenna. We assume no cooperation between the terminals $T_1$ and $T_2$.

### *A. Relaying Protocol and Capacity analysis*

We assume that both $T_1$ and $T_2$ transmit data in the spatial multiplexing mode. Therefore there are $2M$ data streams present in the network. The main idea behind this protocol is the following. We assume that each relay in the network is assigned to one out of the $2M$ data streams, for which it performs MRC in the the receive mode and beamforming in the transmit mode. We have $K$ relays in the network, therefore each data stream is served by $K/2M$ relays. We represent the $M$ data streams going from $T_1$ to $T_2$ by $x_1, x_2, \ldots x_M$ and the $M$ data streams going from $T_2$ to $T_1$ by $u_1, u_2, \ldots u_M$. We partition the set of relays into $2M$ sets, with each set (of size $K/2M$) being responsible for one of the $2M$ data streams. We denote by $\mathcal{X}_l$ and $\mathcal{U}_l$, the set of relays assigned to data stream $x_l$ and $u_l$, $l = 1, 2, \ldots, M$, respectively. This protocol is similar to one given by [13], [14]. With the help of this protocol we have the following Theorem on the achievable rate for coherent two-way relaying.

*Theorem 2:* For coherent two-way relaying between $T_1$ and $T_2$, each equipped with $M$ antennas, via $K$ relays (each with $N$ fixed number of antennas), the total achievable rate scales as $M \log(K) + \mathcal{O}(1)$ as $K \to \infty$, with no cooperation required between $T_1$ and $T_2$.

*Proof:* Due to lack of space we only give a sketch of the proof. From (1), the received signal at $k^{th}$ relay is given by

$$\mathbf{r}_k = \sqrt{\frac{E_k}{M}} \sum_{j=1}^{M} \mathbf{h}_{jk} x_j + \sqrt{\frac{F_k}{M}} \sum_{j=1}^{M} \mathbf{g}_{jk}^{(r)} u_j + \mathbf{n}_k.$$

Let us assume that the relay $k$ is assigned to data stream $x_i$ ($k \in \mathcal{X}_i$) and relay $m$ is assigned to data stream $u_i$ ($m \in \mathcal{U}_i$), then employing MRC at relay $k$ and $m$ for $x_i$ and $u_i$ the processed signal is given by

$$p_k \triangleq \mathbf{h}_{ik}^H \mathbf{r}_k, q_k \triangleq \mathbf{g}_{im}^{(r)H} \mathbf{r}_m$$

respectively. At each of the relays, we do beamforming for a single stream on the forward channel. Therefore the transmitted signal $\mathbf{t}_k$ and $\mathbf{t}_m$ by relay $k$ and $m$ respectively, are given by

$$\mathbf{t}_k = \theta_f p_k \frac{\mathbf{g}_{ki}^H}{||\mathbf{g}_{ki}||}, \quad \mathbf{t}_m = \theta_b q_k \frac{\mathbf{h}_{mi}^{(r)H}}{||\mathbf{h}_{mi}^{(r)}||}$$

where $\theta_f$ and $\theta_b$ are the scaling factors such that the power constraint $\mathcal{E}(\mathbf{t}_k^H \mathbf{t}_k) = N$, is met at each of the relay.

Consider the $T_1$ to $T_2$ link. First we compute the achievable rate on this link. From (2), the received signal at the $i^{th}$ antenna of $T_2$ is given by

$$y_i = \sqrt{\frac{P_k}{N}} \sum_{k=1}^{K} \mathbf{g}_{ki} \mathbf{t}_k + z_i \tag{6}$$

substituting $\mathbf{t}_k$ and separating $y_i$ in terms of the desired signal $x_i$, the interference signal $x_l, \ l \neq i$, the self interference generated by $u_l, l = 1, 2, \ldots M$, the forwarded noise from the relays and the receiver noise,

$$y_i = \mathbf{h}_i^{sig} x_i + \sum_{j=1, j \neq i}^{M} \mathbf{h}_{i,j}^{int} x_j + \sum_{j=1}^{M} \mathbf{h}_j^{selfInt} u_j + N_i$$

An important observation is that, $T_2$ knows $u_i, \ i = 1, 2, \ldots M$ and therefore with knowledge of $\mathbf{h}_{ki}, \mathbf{h}_{ki}^{(r)}, \mathbf{g}_{ki}$ and $\mathbf{g}_{ik}^{(r)}$ $\forall k$, it can cancel the $\sum_{j=1}^{M} \mathbf{h}_j^{selfInt} u_j$ term from the received signal at $i^{th}$ receive antenna. Removing the self interference term from $y_i$

$$y_i' = \mathbf{h}_i^{sig} x_i + \sum_{j=1, j \neq i}^{M} \mathbf{h}_{i,j}^{int} x_j + N_i.$$

Since $T_2$ knows $\{E_k, P_k, F_k, \mathbf{H}_k, \mathbf{G}_k, \mathbf{H}_k^{(r)}, \mathbf{G}_k^{(r)}\}_{k=1}^{K}$ perfectly, the noise $N_i + \sum_{j=1, j \neq i}^{M} \mathbf{h}_{ij}^{int} x_j$ contribution can be shown to be circularly symmetric complex Gaussian. Due to independent decoding of each of the $M$ data streams at each receive antenna of $T_2$ and noise plus interference contribution being circularly symmetric complex Gaussian, using [19], the achievable rate $R_{12}$ on the $T_1 \to T_2$ link is given by

$$R_{12} = \sum_{i=1}^{M} \mathcal{E}_{\{\mathbf{H}_k, \mathbf{G}_k\}_{k=1}^{K}} I_i \tag{7}$$

where $I_i$ is given by

$$I_i = \frac{1}{2} \log \left( 1 + \frac{|\mathbf{h}_i^{sig}|^2}{\sum_{j=1, j \neq i}^{M} |\mathbf{h}_i^{int}|^2 + \sigma^2 (1 + \Delta)} \right) \tag{8}$$

and $\Delta$ is the contribution of the forwarded noise from the relays. By assumption, $E_k$ and $P_k$ are positive and bounded $\forall k$ and with some work it is possible to show that the conditions of (Theorem $1.8D$[21]) are satisfied and hence as $K \to \infty$

$$\frac{\mathbf{h}_i^{sig}}{K/M} - \frac{\mathcal{E}\{\mathbf{h}_i^{sig}\}}{K/M} \xrightarrow{w.p.\ 1} 0$$

$$\frac{\mathbf{h}_i^{int}}{K/M} - \frac{\mathcal{E}\{\mathbf{h}_i^{int}\}}{K/M} \xrightarrow{w.p.\ 1} 0$$

$$\frac{\Delta}{K/M} - \frac{\mathcal{E}\{\Delta\}}{K/M} \xrightarrow{w.p.\ 1} 0$$

The important point to note is that each term in $\mathbf{h}_i^{int}$ involves product of two independent channel coefficients, each with mean $0$, which implies $\mathcal{E} \mathbf{h}_i^{int}\} = 0$. Using this observation, multiplying and dividing the argument of the logarithm in (8) by $(K/M)^2$ and applying (Theorem $1.7$[21])

$$I_i \xrightarrow{w.p.\ 1} \frac{1}{2} \log \left( 1 + \frac{K \delta_i^2}{M \sigma^2 \left( M/K + \tilde{\Delta} \right)} \right) \tag{9}$$

as $K \to \infty$, where

$$\delta_i = \frac{\mathcal{E}\{\mathbf{h}_i^{sig}\}}{K/M}, \ \tilde{\Delta} = \frac{\mathcal{E}\{\Delta\}}{K/M}$$

Hence from (7), the achievable rate from $T_1 \to T_2$ is given by

$$R_{12} \xrightarrow{w.p.\ 1} \frac{1}{2} \sum_{i=1}^{M} \log \left( 1 + \frac{K \delta_i^2}{M \sigma^2 \left( M/K + \tilde{\Delta} \right)} \right)$$

and thus for $K \to \infty$.

$$R_{12} = \frac{M}{2}\log(K) + \mathcal{O}(1).$$

For the achievable rate analysis on the $T_2 \to T_1$ link, consider the received signal $v_i$ at the $i^{th}$ antenna of $T_1$. From (2)

$$v_i = \sqrt{\frac{Q_k}{N}}\sum_{k=1}^{K}\mathbf{h}_{ki}^{(r)}\mathbf{t}_k + w_i$$

Compared to $y_i$ (6), $v_i$ has random variables $Q_k$ in place of $P_k$ and $\mathbf{h}_{ki}^{(r)}$ in place of $\mathbf{g}_{ki}$, however the distribution of $Q_k$ and $\mathbf{h}_{ki}^{(r)}$ is identical to that of $P_k$ and $\mathbf{g}_{ki}$, respectively. Note that the number of relays serving $u_i$ is same as the number of relays serving $x_i,\ i = 1, 2, \ldots M$ and $T_1$ knows $x_i\ \forall i$ and therefore with channel knowledge of $\mathbf{H}_k, \mathbf{H}_k^{(r)}, \mathbf{G}_k, \mathbf{G}_k^{(r)}\}\ \forall k$, can also cancel the self interference. Putting all these facts together, it is clear that the analysis for the achievable rate on the $T_2 \to T_1$ link is exactly the same that of $T_1 \to T_2$ link and for the sake of brevity we do not include it here. We conclude that the achievable rate $R_{21}$ on the $T_2 \to T_1$ link is also given by

$$R_{21} = \frac{M}{2}\log(K) + \mathcal{O}(1).$$

as $K \to \infty$. Therefore the total achievable rate for two way relaying is given by

$$R = R_{12} + R_{21} = M\log(K) + \mathcal{O}(1)$$

as $K \to \infty$. ■

## V. TWO-WAY RELAYING CAPACITY RESULTS AND CONCLUSION

Combining Theorem 1 and 2, the coherent two-way relaying capacity is given by $M\log(K) + \mathcal{O}(1)$, as $K \to \infty$. The $\mathcal{O}(1)$ term in the upper and lower bound can in general be different and hence we characterize the exact capacity upto a $\mathcal{O}(1)$ term. This result also implies that our relaying protocol of MRC and beamforming at each relay, on a per data stream basis, is optimal in the sense of achieving the right capacity scaling. One important observation to be made is that the achievable capacity result in Theorem 2 does not require any cooperation between $T_1$ and $T_2$. This is significant since the upperbound is for some joint encoding between $T_1$ and $T_2$. This is possible because with channel knowledge, both $T_1$ and $T_2$ are able to cancel off the self interference, their own signals generate.

Compared to [13], [14], our results show that by two-way relaying one can remove the $\frac{1}{2}$ rate loss factor on the capacity, which comes from the half-duplex assumption on the terminals and relays. The increase in capacity is primarily due to the fact that with perfect CSI each terminal is able to cancel off the self interference its own transmitted signal generates. We show that with two-way relaying one can remove the $\frac{1}{2}$ rate loss factor in spectral efficiency due to the half-duplex constraint and hence we can get full duplex performance with half duplex nodes.

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