Degrees of Freedom for the MIMO Interference Channel

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***Abstract*— We show that the exact number of spatial degrees of freedom for a two user nondegenerate (full rank channel matrices) MIMO Gaussian interference channel with M_1, M_2 (respectively) antennas at transmitters $1, 2$ and N_1, N_2 antennas at the corresponding receivers, and perfect channel knowledge at all transmitters and receivers, is $\min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\}$. A constructive achievability proof shows that zero forcing is sufficient to achieve all the available degrees of freedom on the two user MIMO interference channel. We also show through an example of a share-and-transmit scheme how the gains of transmitter cooperation may be entirely offset by the cost of enabling that cooperation so that the available degrees of freedom are not increased.**

***Index Terms*— Broadcast, Degrees of Freedom, Interference, MIMO, Multiple Access, Zero Forcing.**

I. Introduction

Multiple input multiple output (MIMO) systems have assumed great importance in recent times because of their remarkably higher capacity compared to single input single output systems. It is well known [1]–[3] that capacity of a point to point (PTP) MIMO system with M inputs and N outputs increases linearly as $\min(M, N)$ at high signal to noise power ratio (SNR). For power and bandwidth limited wireless systems, this opens up another dimension - "space" that can be exploited in a similar way as time and frequency. Similar to time division and frequency division multiplexing, MIMO systems present the possibility of multiplexing signals in space. Spatial dimensions are especially interesting for how they may be limited by distributed processing as well the amount of channel knowledge. Previous work has shown that in the absence of channel knowledge, spatial degrees of freedom are lost [4], [5]. Multiuser systems, with constrained cooperation between inputs/outputs distributed among multiple users, are especially challenging since, unlike PTP case, joint processing is not possible at inputs/outputs. The available spatial degrees of freedom are affected by the inability to jointly process the signals at the distributed inputs and outputs. The two user interference channel with single antennas at all nodes is considered by Host-Madsen [6], [7]. It is shown that the maximum multiplexing gain is only equal to one even if cooperation between the two transmitters or the two receivers is allowed via a noisy communication link. Nosratinia and Host-Madsen [8] show that even if communication links are introduced between the two transmitters as well as between the two receivers the highest multiplexing gain achievable is equal to one. These results are somewhat surprising as it can be shown that with ideal cooperation between transmitters (broadcast channel) or with ideal cooperation between receivers (multiple access channel) the maximum multiplexing gain is equal to 2.

In this paper, we focus on the two user (M_1, N_1, M_2, N_2) MIMO interference channel where transmitter 1 with M_1 antennas has a message for receiver 1 with N_1 antennas, and transmitter 2 with M_2 antennas has a message for receiver 2 with N_2 antennas. We develop a MIMO multiple access channel (MAC) outerbound on the sum capacity of this MIMO interference channel. The outerbound is used to prove a converse result for the maximum number of degrees of freedom. We also provide a constructive proof of achievability of the degrees of freedom based on zero forcing. We show that the innerbound and the outerbound are tight, thereby establishing the precise number of degrees of freedom on the MIMO interference channel as $\min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\}$. We also consider a simple cooperative scheme to understand why transmitter cooperation may not increase degrees of freedom. Through this simple scheme, we are able to show how the benefits of cooperation can be completely offset by the cost of enabling it.

II. Degrees of Freedom Measure

We assume that channel state is fixed and perfectly known at all transmitters and receivers. Also, we assume that the channel matrices are sampled from a rich scattering environment. Therefore we can ignore the measure zero event that some channel matrices are rank deficient. It is well known that the capacity of a *scalar* additive white Gaussian noise (AWGN) channel scales as $\log(SNR)$ at high SNR. On the other hand, for a single user MIMO channel with M inputs and N outputs, the capacity growth rate can be shown to be $\min(M, N)\log(SNR)$ at high SNR. This motivates the natural definition of spatial degrees of freedom as:

$$\eta \triangleq \lim_{\rho \to \infty} \frac{C_\Sigma(\rho)}{\log(\rho)}, \tag{1}$$

where $C_\Sigma(\rho)$ is the sum capacity (just capacity in case of PTP channels) at SNR ρ. In other words, degrees of freedom η represent the maximum *multiplexing gain* [3] of the generalized MIMO system. For PTP case, $\min(M, N)$ degrees of freedom are easily seen to correspond to the parallel channels that can be separated using the singular value decomposition (SVD) of

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the channel matrix, involving joint processing at the M inputs and N outputs, i.e.

$$\eta(\text{PTP}) \quad = \quad \min(M, N) \qquad (2)$$

A. The Multiple Access Channel

The MAC channel is an example of a MIMO system where cooperation is allowed only between the channel outputs. Let the MAC consist of N outputs controlled by the same receiver and 2 users, each controlling M_1 and M_2 inputs for a total of $M = M_1 + M_2$ inputs. For the MAC, the available degrees of freedom are the same as with perfect cooperation between all users.

$$\eta(\text{MAC}) = \eta(\text{PTP}) = \min(M_1 + M_2, N). \qquad (3)$$

The converse is straightforward because, for the same number of inputs and outputs, $\eta(\text{MAC}) \leq \eta(\text{PTP}) = \min(M_1 + M_2, N)$. In other words, the lack of cooperation at the inputs can not increase degrees of freedom. For achievability, it is interesting to note that zero forcing (ZF), which is normally a suboptimal strategy, is easily seen to be sufficient to utilize all degrees of freedom.

B. The Broadcast Channel

The BC channel is an example of a MIMO system where cooperation is allowed only between the channel inputs. Let the BC consist of M inputs controlled by the same transmitter and 2 users, each controlling N_1 and N_2 outputs for a total of $N = N_1 + N_2$ outputs. In a similar fashion as the MAC, it is easy to show that by ZF at the BC transmitter, $\min(M, N)$ parallel channels can be created, so that the total degrees of freedom are the same as with perfect cooperation between all the users.

$$\eta(\text{BC}) = \eta(\text{MAC}) = \eta(\text{PTP}) = \min(M, N). \qquad (4)$$

III. Interference Channel

Consider an $(M_1, N_1), (M_2, N_2)$ interference channel with two transmitters T_1 and T_2, and two receivers R_1 and R_2, where T_1 has a message for R_1 only and T_2 has a message for R_2 only. T_1 and T_2 have M_1 and M_2 antennas respectively. R_1 and R_2 have N_1 and N_2 antennas respectively. The interference channel is characterized by the following input output relationships:

$$Y^{(1)} \quad = \quad H^{(1)}X^{(1)} + Z^{(1)}X^{(2)} + W^{(1)} \qquad (5)$$
$$Y^{(2)} \quad = \quad H^{(2)}X^{(2)} + Z^{(2)}X^{(1)} + W^{(2)}, \qquad (6)$$

where we denote the $N_1 \times M_1$ channel matrix between T_1 and R_1 by $H^{(1)}$, the $N_2 \times M_2$ channel matrix between T_2 and R_2 by $H^{(2)}$, the $N_2 \times M_1$ channel matrix between T_1 and R_2 by $Z^{(2)}$, and the $N_1 \times M_2$ channel matrix T_2 and R_1 by $Z^{(1)}$. $X^{(1)}, X^{(2)}$ are the M_1 and M_2 dimensional inputs vectors, $Y^{(1)}, Y^{(2)}$ are the N_1 and N_2 dimensional output vectors, and $W^{(1)}, W^{(2)}$ are the N_1 and N_2 dimensional additive white Gaussian noise (AWGN) vectors, respectively. As mentioned before, we assume that the channels are non-degenerate, i.e., all channel matrices are full rank. Figure 1 shows an illustration of this interference channel. Without loss of generality we arrange the links so that link 1 always has the most number of antennas either at its transmitter or receiver, i.e. $\max(M_1, N_1) \geq \max(M_2, N_2)$.

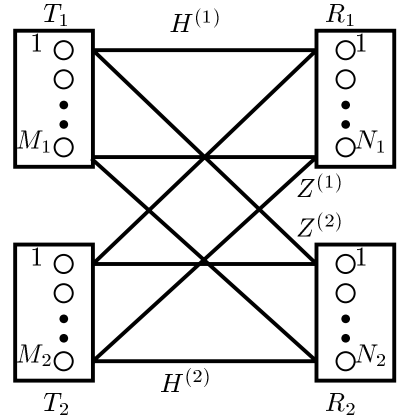

Fig. 1. (M_1,N_1) , (M_2,N_2) Interference channel

A. Achievability: Innerbound on the Degrees of Freedom

For the $(M_1, N_1), (M_2, N_2)$ interference channel we prove the following innerbound on the available degrees of freedom.

Lemma 1:

$$\begin{aligned}\eta(\text{INT}) \quad &\geq \quad \min(M_1, N_1) \\ &+ \quad \min(M_2 - N_1, N_2)^+ \ 1(M_1 > N_1) \\ &+ \quad \min(M_2, N_2 - M_1)^+ \ 1(M_1 < N_1), \end{aligned} \qquad (7)$$

where $1(.)$ is the indicator function and $(x)^+ = \max(0, x)$.

Sketch of Achievability Proof: According to our model, either $M_1 \geq N_1, M_2, N_2$ or $N_1 \geq M_1, M_2, N_2$. We explain the zero forcing based constructive achievability argument for the case when $M_1 \geq N_1, M_2, N_2$. The case with $N_1 \geq M_1, M_2, N_2$ follows similarly and is omitted to avoid repetition.

Based on (7), when $M_1 \geq N_1, M_2, N_2$, we need to show the achievability of $N_1 + \min(M_2 - N_1, N_2)^+ \ 1(M_1 > N_1)$ degrees of freedom. If either $M_1 = N_1$ or $M_2 \leq N_1$ then we need to show the achievability of only N_1 degrees of freedom which can be trivially achieved by only allowing communication between T_1 and R_1. Therefore we consider the remaining case of $M_1 > N_1$ and $M_2 > N_1$. In this case we need to show the achievability of $N_1 + \min(M_2 - N_1, N_2)$ degrees of freedom. Fig. 2 illlustrates the scheme described in the remainder of this section with the example of an interference channel with $M_1 = 5, M_2 = 4, N_1 = 3, N_2 = 3$ where a total of 4 degrees of freedom are achieved.

Step 1: Let the singular value decomposition (SVD), $Z^{(1)} = U^{(1)}\Lambda^{(1)}V^{(1)\dagger}$ and $Z^{(2)} = U^{(2)}\Lambda^{(2)}V^{(2)\dagger}$, where $U^{(1)}$, $V^{(1)}$, $U^{(2)}$, $V^{(2)}$ are $N_1 \times N_1$, $M_2 \times M_2$, $N_2 \times N_2$, and $M_1 \times M_1$ unitary matrices, respectively. $\Lambda^{(1)}$, $\Lambda^{(2)}$ are $N_1 \times M_2$ and $N_2 \times M_1$ matrices with singular values of $Z^{(1)}$, $Z^{(2)}$ respectively on the main diagonal and zeros elsewhere. Using the standard MIMO SVD approach we absorb the unitary matrices into the corresponding input and output vectors to obtain:

$$Y^{(1)'} = H^{(1)'}X^{(1)'} + \Lambda^{(1)}X^{(2)'} + W^{(1)'} \quad (8)$$
$$Y^{(2)'} = H^{(2)'}X^{(2)'} + \Lambda^{(2)}X^{(1)'} + W^{(2)'} \quad (9)$$

where $Y^{(1)'} = U^{(1)\dagger}Y^{(1)}$, $Y^{(2)'} = U^{(2)\dagger}Y^{(2)}$, $X^{(1)'} = V^{(2)\dagger}X^{(1)}$, $X^{(2)'} = V^{(1)\dagger}X^{(2)}$, $W^{(1)'} = U^{(1)\dagger}W^{(1)}$, $W^{(2)'} = U^{(2)\dagger}W^{(2)}$, $H^{(1)'} = U^{(1)\dagger}H^{(1)}V^{(2)}$ and $H^{(2)'} = U^{(2)\dagger}H^{(2)}V^{(1)}$. In particular note that only the first N_1 columns of $\Lambda^{(1)}$ are non-zero.

$$\Lambda^{(1)} = \left[\text{Diag}(\lambda_1^{(1)}, \cdots, \lambda_{N_1}^{(1)}) \;\; \mathbf{0}_{N_1\times(M_2-N_1)}\right] \quad (10)$$

Therefore, only the inputs $X_1^{(2)'}, X_2^{(2)'}, \cdots, X_{N_1}^{(2)'}$ present interference at R_1 from T_2. Similarly, only the inputs $X_1^{(1)'}, X_2^{(1)'}, \cdots, X_{N_2}^{(1)'}$ present interference at R_2 from T_1. In Fig. 2 the bold channels represent the interference paths after the diagonalization achieved through the SVD as there are $\min(5,3) = 3$ parallel paths from T_1 to R_2 and $\min(4,3) = 3$ parallel paths from T_2 to R_1.

Step 3: At transmitter T_1 we set inputs $X_1^{(1)'}, X_2^{(1)'}, \cdots, X_{M_1-N_1}^{(1)'}$ to zero, i.e. we do not transmit on these inputs. This leaves N_1 available inputs $X_{M_1-N_1+1}^{(1)'}, \cdots, X_{M_1}^{(1)'}$ at T_1. For the example of Fig. 2 the $M_1 - N_1 = 2$ transmit antennas indicated by white circles have their inputs set to zero and the dark circles indicate the 3 available at T_1.

Step 4: At transmitter T_2 we set inputs $X_1^{(2)'}, X_2^{(2)'}, \cdots, X_{N_1}^{(2)'}$ to zero, i.e. we do not transmit on these inputs. This leaves $M_2 - N_1$ available inputs $X_{N_1+1}^{(2)'}, \cdots, X_{M_2}^{(2)'}$ at T_2. Fig. 2 illustrates this step as the 3 unused inputs are indicated by white circles and the remaining $M_2 - N_1 = 1$ input by a dark circle.

Step 5: The previous step eliminates any interference from T_2 to R_1 since all the interfering inputs have been set to 0. Therefore, communication between T_1 and R_1 takes place over an $N_1 \times N_1$ MIMO channel with no interference from T_2. N_1 degrees of freedom are achieved through this communication.

Step 6: At receiver R_2 we consider only outputs $Y_1^{[2]'}, Y_2^{[2]'}, \cdots, Y_{\min(M_1-N_1,N_2)}^{[2]'}$ and discard the rest. Note that because of Step 3 these outputs do not contain any interference from T_1.

Step 7: From Step 4, we have $M_2 - N_1$ available inputs at T_2. From Step 6, we have $\min(M_1 - N_1, N_2)$ outputs at R_2 with no interference from T_1. Therefore the communication between T_2 and R_2 takes place over a MIMO channel with $\min(M_2 - N_1, \min(M_1 - N_1, N_2)) = \min(M_2 - N_1, N_2)$ degrees of freedom.

Combining Steps 5 and 7 we have established the achievability of the required total of $N_1 + \min(M_2 - N_1, N_2)$ degrees of freedom. Fig. 2 illustrates the proof with white circles indicating discarded inputs and outputs and black circles indicating the inputs and outputs used for the achievability scheme described above.

B. Converse: Outerbound on the Degrees of Freedom

For the $(M_1, N_1), (M_2, N_2)$ interference channel we prove the following outerbound on the available degrees of freedom.

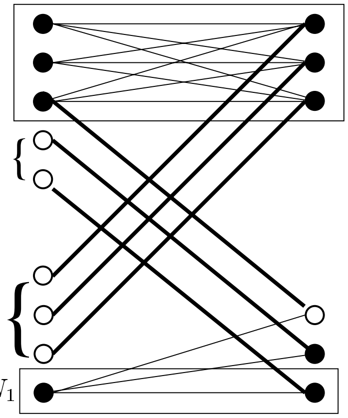

Fig. 2. Achievability proof for (M_1,N_1) , (M_2,N_2) Interference channel when $M_1 \geq M_2, N_1, N_2$

Lemma 2:

$$\eta(\text{INT}) \leq \min\{M_1+M_2, N_1+N_2, \max(M_1,N_2), \max(M_2,N_1)\}$$

To start with, notice that a trivial outerbound is obtained from the PTP case, i.e. $\eta(\text{INT}) \leq \min(M_1 + M_2, N_1 + N_2)$. Indeed this outerbound coincides with the innerbound when either $\min(M_1, M_2) \geq N_1 + N_2$ or $\min(N_1, N_2) \geq M_1 + M_2$. In general, while the capacity region of the interference channel is not known even with single antennas at all nodes, various outerbounds have been obtained [9]–[11] that have been useful in finding the capacity region in some special cases [12], [13]. Most of the existing outerbounds are for single antenna systems.

For our purpose, we develop a genie based outerbound for MIMO interference channel where the number of antennas at either receiver is $\geq$ the number of transmit antennas at the interfering transmitter, i.e. either $N_1 \geq M_2$ or $N_2 \geq M_1$. This outerbound is the key to the tight converse needed to establish the number of degrees of freedom. Note that for this section, since we do not need the assumption that $\max(M_1, N_1) \geq \max(M_2, N_2)$, the proof for the cases $N_1 \geq M_2$ or $N_2 \geq M_1$ is identical.

Theorem 1: For the $(M_1, N_1), (M_2, N_2)$ interference channel with $N_1 \geq M_2$, the sum capacity is bounded above by that of the corresponding (M_1, M_2, N_1) MAC channel with additive noise $W^{(1)} \sim \mathcal{N}(0, I_{N_1})$ modified to $W^{(1)'} \sim \mathcal{N}(0, K')$ where

$$K' = I_{N_1} - Z^{(1)}\left(Z^{(1)\dagger}Z^{(1)}\right)^{-1}Z^{(1)\dagger} + \alpha Z^{(1)}Z^{(1)\dagger},$$
$$\alpha = \min\left(\frac{1}{\sigma^2_{\max}(Z^{(1)})}, \frac{1}{\sigma^2_{\max}(H^{(2)})}\right).$$

where $\sigma_{\max}(A)$ represents the principal singular value of matrix A.

Proof:

Let us define

$$\begin{aligned} W_a^{(1)} &\sim \mathcal{N}\left(0, I_{N_1} - Z^{(1)}\left(Z^{(1)\dagger}Z^{(1)}\right)^{-1}Z^{(1)\dagger}\right) \\ W_b^{(1)} &\sim \mathcal{N}\left(0, Z^{(1)}\left(Z^{(1)\dagger}Z^{(1)}\right)^{-1}Z^{(1)\dagger} - \alpha Z^{(1)}Z^{(1)^\dagger}\right) \\ W_c^{(1)} &\sim \mathcal{N}\left(0, \alpha Z^{(1)}Z^{(1)^\dagger}\right), \end{aligned}$$

as three $N_1 \times 1$ jointly Gaussian and mutually independent random vectors. The positive semidefinite property of the respective covariance matrices is easily established from the definition of α.

Without loss of generality we assume

$$\begin{aligned} W^{(1)} &= W_a^{(1)} + W_b^{(1)} + W_c^{(1)} \\ W^{(1)'} &= W_a^{(1)} + W_c^{(1)} \end{aligned}$$

Since a part of the proof is similar to the corresponding proof for the single antenna case, we will summarize the common steps, and emphasize only the part that is unique to MIMO interference channel. Consider any achievable scheme for any rate point within the capacity region of the interference channel, so that R_1 and R_2 can correctly decode their intended messages from their received signals with sufficiently high probability.

Step 1: We replace the original additive noise $W^{(1)}$ at R_1 with $W^{(1)'}$ as defined in Theorem 1. We argue that this does not make the capacity region smaller because the original noise statistics can easily be obtained by locally generating and adding noise $W_b^{(1)}$ at R_1. Therefore, since R_1 was originally capable of decoding its intended message with noise $W^{(1)}$, it is still capable of decoding its intended message with $W^{(1)'}$.

Step 2: Suppose that a genie provides R_2 with side information containing the entire codeword $X^{(1)}$. Since $X^{(2)}$ is independent of $X^{(1)}$, R_2 simply subtracts out the interference from its received signal. Thus, the channel $Z^{(2)}$ can be eliminated without making the capacity region smaller.

Step 3: By our assumption, R_1 can decode its own message and therefore it can subtract $X^{(1)}$ from its own received signal as well. In this manner, after the interfering signals have been subtracted out we have

$$\begin{aligned} Y^{(1)} &= Z^{(1)}X^{(2)} + W^{(1)'}, \\ Y^{(2)} &= H^{(2)}X^{(2)} + W^{(2)}. \end{aligned}$$

To complete the proof we need to show that if R_2 can decode $X^{(2)}$ then so can R_1. This would imply that R_1 can decode both messages, hence giving us the MAC outer bound.

Step 4: Without loss of generality, let us perform SVD $H^{(2)} = F^{(2)}\Sigma^{(2)}G^{(2)\dagger}$ on the channel between T_2 and R_2. This is a lossless operation that leads to:

$$Y^{(2)\text{new}} = X^{(2)\text{new}} + W^{(2)''}, \tag{11}$$

where $X^{(2)\text{new}} = G^{(2)\dagger}X^{(2)}$ and $W^{(2)''}$ is additive noise that consists of independent zero mean complex Gaussian random variables with variances $\frac{1}{\sigma_i^2(H^{(2)})}$ and $\sigma_i(H^{(2)})$ are the singular values of $H^{(2)}$. Note that we have dropped dimensions that correspond to zero channel gains as these channels are useless for R_2.

Step 5: Next, we show that R_1 can obtain a stronger channel to $X^{(2)\text{new}}$ so that if R_2 can decode it, so can R_1. To this end, let R_1 use ZF to obtain:

$$\begin{aligned} Y^{(1)\text{new}} &= X^{(2)\text{new}} + V^{(2)}\left(Z^{(1)\dagger}Z^{(1)}\right)^{-1}Z^{(1)\dagger}W^{(1)'}, \\ &= X^{(2)\text{new}} + W^{(1)''} \end{aligned}$$

where $W^{(1)''}$ is a vector of AWGN with i.i.d. elements and variance α.

Now both R_1 and R_2 have a diagonal channel with input $X^{(2)\text{new}}$ and uncorrelated additive white noise components on each diagonal channel. Moreover, the strongest channel for R_2 has noise $\frac{1}{\sigma_{\max}^2(H^{(2)})}$. However the noise on any channel for R_1 is only α which is smaller. Thus, we argue once again that R_1 can locally generate noise and add it to its received signal to create a statistically equivalent noise signal as seen by R_2. In other words, R_1 has a less noisy channel to T_2 and therefore can decode any signal that R_2 can. Since R_1 can decode T_1's message by assumption, we have the MAC outerbound. ∎

The previous theorem leads directly to the following corollary:

Corollary 1: For the $(M_1, N_1), (M_2, N_2)$ interference channel the number of spatial degrees of freedom $\eta(\text{INT}) \leq \max(M_2, N_1)$.

Proof: If $M_2 \leq N_1$ the sum capacity of the interference channel is upperbounded by the multiple access channel with N_1 receive antennas. Therefore, for $M_2 \leq N_1$ we must have $\eta(\text{INT}) \leq N_1$. Now, if $M_2 > N_1$, then let us add more antennas to receiver 1 so that it has a total of M_2 receive antennas. Additional receive antennas cannot hurt, so the converse argument is not violated. However, with M_2 receive antennas at receiver 1, once again the multiple access upperbound applies to the new interference channel. The number of degrees of freedom is therefore upperbounded as $\eta(\text{INT}) \leq M_2$ when $M_2 > N_1$. Combining the two cases, we have the result of the corollary $\eta(\text{INT}) \leq \max(M_2, N_1)$. ∎

Simply by switching the arguments to user 2 instead of user 1, Corollary 1 leads to another upperbound: $\eta(\text{INT}) \leq \max(M_1, N_2)$ that holds for all M_1, M_2, N_1, N_2. Combining the two upperbounds of the Corollary and the trivial PTP upperbounds we have the converse result.

Finally we show that the achievable innerbound and the converse outerbound are always tight. The following theorem presents the main result of the paper.

Theorem 2: For the $(M_1, N_1), (M_2, N_2)$ interference channel the number of spatial degrees of freedom

$$\begin{aligned} \eta(\text{INT}) &= \min(M_1, N_1) \\ &+ \min(M_2 - N_1, N_2)^+ \ 1(M_1 > N_1) \\ &+ \min(M_2, N_2 - M_1)^+ \ 1(M_1 < N_1) \\ &= \min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\} \end{aligned}$$

Proof: The proof is found by verifying directly that the number of degrees of freedom obtained from the inner and outerbounds always match. The resulting number D from the $\eta(\text{INT})$ inner and outerbounds is listed for all cases in Table I.

$M_1>(M_2,N_1,N_2)$			$N_1\geq(M_1,M_2,N_2)$		
$N_1\geq M_2$	$N_1<M_2$		$N_2\geq M_1+M_2$	$N_2<M_1+M_2$	
$D=N_1$	$M_2\leq N_1+N_2$	$M_2>N_1+N_2$	$D=M_1+M_2$	$N_2\geq M_1$	$N_2<M_1$
	$D=M_2$	$D=N_1+N_2$		$D=N_2$	$D=M_1$

TABLE I

THE SAME NUMBER OF DEGREES OF FREEDOM ARE OBTAINED FROM THE UPPERBOUND AND THE LOWERBOUND IN ALL CASES

(M_1,N_1)	(M_2,N_2)	$\eta(INT)$
$(1,1)$	$(1,1)$	1
$(1,2)$	$(1,2)$	2
$(2,1)$	$(2,1)$	2
$(1,2)$	$(2,1)$	1
$(3,2)$	$(2,3)$	2
$(2,3)$	$(2,3)$	3
$(2,3)$	$(1,3)$	3
$(2,2)$	$(3,2)$	2
(n,m)	(m,n)	$\min(m,n)$
(m,n)	(m,n)	$\min(2m,n)(n\geq m)$

TABLE II

DEGREES OF FREEDOM OF MIMO INTERFERENCE CHANNELS FOR VARIOUS M_1, M_2, N_1, N_2.

∎

Thus we have the exact number of degrees of freedom for all possible M_1, M_2, N_1, N_2. Some examples are provided in Table II. A couple of observations can be made about the spatial degrees of freedom. First, there is a reciprocity in that $\eta(INT)$ is unaffected if M_1 and M_2 are switched with N_1 and N_2 respectively. In other words, the degrees of freedom are unaffected if the directions of the messages are reversed. However, notice that $\eta(INT)$ may change if only M_1 and N_1 are switched while M_2 and N_2 are not switched. Finally from the constructive achievability proof one can see that the available degrees of freedom can be divided among the two users in all possible ways so that the sum is $\eta(INT)$ and the individual degree of freedom allocations are within the individual maxima of $\max(M_1, N_1)$ for user 1 and $\max(M_2, N_2)$ for user 2.

IV. EFFECT OF TRANSMIT COOPERATION ON THE NUMBER OF DEGREES OF FREEDOM

Comparing the interference channel and the BC channel obtained by full cooperation between the transmitters, it is clear that the available degrees of freedom are severely limited by the lack of transmitter cooperation in the interference channel. As an example, consider the interference channel with $(M_1, N_1) = (n, 1)$ and $(M_2, N_2) = (1, n)$. From the preceding section we know there is only one available degree of freedom in this channel. However, if full cooperation between the transmitters is possible the resulting BC channel has $(M, N_1, N_2) = (n + 1, 1, n)$. The number of degrees of freedom is now $n+1$. Therefore, transmitter cooperation would seem highly desirable. Rather surprisingly, it has been shown recently [6] that for the $(1, 1), (1, 1)$ interference channel, allowing the transmitters to cooperate through a wireless link between them (even with full duplex operation), does not increase degrees of freedom. For MIMO interference channels, as suggested by the example above, the potential benefits of cooperation are even stronger and it is not known if transmitter cooperation can increase degrees of freedom. The capacity results of [6] do not seem to allow direct extensions to MIMO interference channels.

To gain insights into the cost and benefits of cooperation in a MIMO interference channel, we consider a specific scheme where transmitters first share their information in a full duplex mode as a MIMO channel (step 1) and subsequently transmit together as BC channel. We will refer to this scheme as the share-and-transmit scheme.

A. *Degrees of Freedom with Share-and-Transmit*

Consider an $(M, N), (M, N)$ interference channel ($M \leq N$). Also assume that each transmitter is sending information with rate R. Note that while we make the preceding simplifying assumptions for simplicity of exposition, the following analysis and the main result extend directly to the general case of unequal number of antennas and unequal rates.

From (7), we know that the number of degrees of freedom for this interefernce channel with no transmitter cooperation is $\min(M, N) + \min(M, N - M) = \min(2M, N)$. For the share-and-transmit scheme, we compute degrees of freedom as follows. We first find the capacity of the sharing link C_s and the capacity of transmission C_t. Then, we find the total capacity of the system C by evaluating the total amount of data transmitted divided by the total time it requires to transmit this data, i.e.

$$C = \frac{2R}{\frac{R}{C_s} + \frac{2R}{C_t}} \quad . \tag{12}$$

Dividing by log(SNR) where SNR is large, we obtain the total number of degrees of freedom as

$$\lim_{SNR\to\infty} \frac{C}{\log SNR} = \frac{2}{\frac{1}{DOF(sharing)} + \frac{2}{DOF(transmit)}} \quad . \tag{13}$$

The number of degrees of freedom for the sharing link is that of MIMO PTP channel with M transmit and receive antennas $= min(M, M) = M$. After transmitters share their information, they can fully cooperate as a $(2M, N, N)$ BC channel. The number of degrees of freedom for this channel is $\min(2M, 2N) = 2\min(M, N)$. Therefore (13), which gives the total number of degrees of freedom for the share-and-transmit scheme, becomes $\frac{2M\min(M,N)}{M+\min(M,N)} = M$. Note that,

$$M + \min(M, N - M)^+ \geq M. \tag{14}$$

Therefore, we conclude that (for this specific scheme) transmitter cooperation in the high SNR regime does not provide any advantage to the number of degrees of freedom in the MIMO interference channel.

V. Conclusions

We investigate the degrees of freedom for the MIMO interference channel. The distributed nature of the antennas significantly limits degrees of freedom. For an interference channel with a total of N transmit antennas and a total of N receive antennas, the available number of degrees of freedom can vary from N to 1 based on how the antennas are distributed among the two transmitters and receivers. Through an example of a share-and-transmit scheme, we show how the gains of transmitter cooperation can be entirely offset by the cost of enabling that cooperation so that the available degrees of freedom are not increased. Our result is in a sense a negative result, because similar to [7] it shows that on the MIMO interference channel there is nothing beyond zero forcing as far as spatial multiplexing is concerned.

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