

# Duality and Rate Optimization for Multiple Access and Broadcast Channels with Amplify-and-Forward Relays

Syed A. Jafar, Krishna S. Gomadam and Chiachi Huang

## Abstract

We consider multi-hop multiple access (MAC) and broadcast channels (BC) where communication takes place with the assistance of relays that amplify and forward (AF) their received signals. For a two hop parallel AF relay MAC, assuming a sum power constraint across all relays we characterize optimal relay amplification factors and the resulting optimal rate regions. We find the maximum sum rate and the maximum rate for each user in closed form and express the optimal rate pair  $(R_1, R_2)$  that maximizes  $\mu_1 R_1 + \mu_2 R_2$  as the solution of a pair of simultaneous equations. We find that the parallel AF relay MAC with total transmit power of the two users  $P_1 + P_2 = P$  and total relay power  $P_R$  is the dual of the parallel AF relay BC where the MAC source nodes become the BC destination nodes, the MAC destination node becomes the BC source node, the dual BC source transmit power is  $P_R$  and the total transmit power of the AF relays is  $P$ . The duality means that the rate region of the AF relay MAC with a sum power constraint  $P$  on the transmitters is the same as that of the dual BC. The duality relationship is found to be useful in characterizing the rate region of the AF relay BC as the union of MAC rate regions. The duality is established for distributed multiple antenna AF relay nodes and multiple (more than 2) hops as well.

## Index Terms

Amplify-and-Forward, Broadcast, Capacity, Duality, MIMO, Multiple Access.

S. Jafar, K. Gomadam and C. Huang are with the Center of Pervasive Communications and Computing (CPCC), Department of Electrical Engineering and Computer Science, University of California Irvine, Irvine, CA, 92697 USA e-mail: syed@uci.edu, kgomadam@uci.edu, chiachih@uci.edu. This work was supported in part by the National Science Foundation under CAREER Grant 0546860 and by DARPA under ITMANET Grant UTA06-793. This work was presented in part at the 2007 IEEE Personal Indoor and Mobile Radio Communications Conference.

The potential for significant capacity improvements has sparked much interest in cooperative wireless communication over multiple hops with the assistance of intermediate relay nodes. For example, recent work has shown remarkable throughput benefits of employing fixed relays (wireless extension points) in cellular systems and wireless mesh backhaul networks [1]–[3]. Various relay strategies have been studied in literature, primarily for point to point communications, i.e., with a single source and a single destination [4]. These strategies include amplify-and-forward (AF) [5], [6], where the relay sends a scaled version of its received signal to the destination; demodulate-and-forward (DemF) [6], where the relay demodulates individual symbols and retransmits those symbols without regard to an outer code; decode-and-forward (DecF) [7] where the relay decodes the entire message, re-encodes it and sends it to the destination; compress-and-forward (CF) [8], [9] where the relay sends a quantized version of its received signal; and estimate and forward (EF) where the relay sends a soft estimate of its received symbol to the destination. Coherent amplify and forward relay networks are especially interesting as they take advantage of the broadcast nature of the wireless medium and the coherent combining of signals that is specific to wireless networks. Understanding AF relay networks offers insights into the role of soft information in a multihop network [10]. Amplify and forward strategy is also interesting from a practical standpoint as it requires the relays to only scale their received symbols. Thus the complexity and cost of relaying, always an issue in designing cooperative networks, is minimal for AF relay networks. In addition to its simplicity AF is known to be the optimal relay strategy in many interesting cases [11]–[15].

AF relay optimization for dual hop communications has been the focus of much recent research. [16]–[18] consider the case of orthogonal relay transmissions. While orthogonal relay schemes are attractive for wideband communications, Maric and Yates [19] and Nabar et. al. [20] have shown that for amplify and forward relays, shared bandwidth transmission schemes can provide higher capacity. Larsson [21] and Maric and Yates [19] also find closed form solutions for the relay amplification factor and the point to point AF relay channel capacity with shared band transmission. Optimum and near optimum power allocation schemes for single branch multi-hop relay networks have been considered in [22].

While much of the work on relay networks has focused on point to point communication, multiuser relay networks are increasingly gaining attention as well [3], [9], [23]–[25]. In [26] gains for AF relays in a multiuser parallel network are determined to achieve a joint minimization of the MMSE of all the source signals at the destination. Tang et. al. [3] consider a MIMO relay broadcast channel, where a multiple antenna transmitter sends data to multiple users via a relay with multiple antennas over two hops. They find different algorithms for computing the transmit precoder, relay linear processing matrix and the sum rate under the assumption of zero-forcing dirty paper coding and Gaussian signals. Capacity with cooperative relays has been explored for the multicast problem by Maric and Yates [27], [28], for the broadcast problem by Liang and Veeravalli [25], and for the mixed multiple access and broadcast problem by Host-Madsen [29]. Maric and Yates explore an accumulative multicast strategy where nodes

collect energy from previous transmissions, while Liang and Veeravalli [25] and Host-Madsen [29] address<sup>3</sup> the general question of optimal relay functionality which may not be an amplify and forward scheme. Azarian et. al. [30] explore the diversity-multiplexing tradeoff in half-duplex, cooperative multiple access and broadcast scenarios for various AF and DecF protocols. With channel knowledge available only to the receiving nodes, the diversity multiplexing tradeoff for AF is shown to be dominated by DecF. In the absence of channel knowledge at the transmitters, AF protocols cannot utilize the array gain from coherent combining of signals at the receiver and therefore nothing is to be gained from more than one relay transmitting the same symbol simultaneously. The results of [19], [31] have shown that if channel knowledge is available at the transmitters as well as the receivers then AF relay networks can benefit significantly from coherent combining of simultaneous transmissions. In this paper we explore this distributed array gain for multiple access and broadcast communications using amplify and forward relay nodes.

## II. OVERVIEW

We first state our general assumptions, followed by a summary of the main results.

*Assumptions:*

- 1) We focus primarily on two hop multiple access and broadcast communication via parallel AF relay links where no direct link exists between the source(s) and destination(s). The absence of a direct link between the source and the destination could represent a half-duplex system where the source transmits on frequency  $f_1$ , all the relays receive on frequency  $f_1$  and transmit on frequency  $f_2$  and the destination receives on frequency  $f_2$ . Our system model is also applicable to a full duplex system where the distance between the source and the destination is large enough for the direct link strength to be negligible. Note that propagation path loss is the main reason for employing relay repeaters in practice.
- 2) While the hops are mutually orthogonal as explained above, we do not assume orthogonality for the signals transmitted by different relays over the same hop. In other words, at each hop, we allow superposition of simultaneous transmissions by the relay nodes so there is the possibility of coherent combining. Using the terminology of [19], the relaying scheme for each hop is a *shared bandwidth* scheme.
- 3) We allow centralized network optimization based on global channel knowledge. This is especially relevant for fixed wireless networks with centralized operation, such as the mesh network of fixed wireless extension points used for the cellular backhaul communications in [2]. With slow channel variation there is enough time to percolate channel knowledge and centralized optimal control decisions through the network. We allow arbitrary channel coefficients with the assumption that the channel coefficients are held fixed throughout the duration of communication. Additive white Gaussian noise (AWGN) with zero mean and unit variance is assumed for each channel.

- 4) We assume a sum power constraint across all relay nodes at each hop. The sum power constraint<sup>4</sup> is interesting for two reasons. First, it provides useful insights into optimal resource allocation for cooperating nodes. For example, with different channel strengths associated with different relays it is intuitive that the benefits of cooperation are optimized if relays with stronger channels participate to a greater extent than relays with weaker channels. Optimal resource allocation across relays is a challenging problem especially when the relays serve multiple users simultaneously, as in multiple access or broadcast channels. The sum power constraint across relays automatically addresses this resource allocation problem. The second reason is that the sum power constraint enables powerful duality relationships that are also of fundamental interest.
- 5) Finally, throughout this paper we consider real signals, noise, channel and relay amplification factors. However, many of the results apply directly to the complex case as well. The exceptions will be pointed out in the conclusions section.

*Results:*

In this work, we pursue two related objectives. The first is to investigate the rate optimal relay phase and power selection for two hop multiple access and broadcast channels with AF relays. With a sum power constraint on all the relays, we characterize the optimal relay amplification factor for the parallel AF relay MAC to maximize any weighted sum of users' rates. We obtain the maximum sum rate and the maximum individual rate for each user in closed form and present a pair of simultaneous equations whose numerical solution yields the optimal rate pair  $(R_1, R_2)$  that maximizes  $\mu_1 R_1 + \mu_2 R_2$  for any  $\mu_1, \mu_2 \geq 0$ . The optimal rate region of the parallel AF relay BC is obtained as the union of the relay MAC optimal rate regions based on a duality relationship described next.

The second objective of this work is to identify duality relationships in AF relay networks. We obtain a general duality result for multi-hop multiple access and broadcast channels where each hop may consist of parallel AF relays and the relays may be equipped with multiple antennas. For the two hop case, we show that the multiple access channel with total transmit power of all users equal to  $P$  and total relay transmit power  $P_R$  is the dual of the BC obtained when the destination becomes the transmitter and the transmitters become the receivers, and the powers are switched as well, i.e. in the dual BC, the transmit power is  $P_R$  and the total relay power is  $P$ . In general, for multi hop AF relay networks with parallel AF relays at each hop, and with possibly multiple antennas at the relays, the duality result holds when each link is associated with the same transmit power in the original MAC and the dual BC channels.

### III. SINGLE USER AF PARALLEL RELAY CHANNEL

We start with the simple illustrative example of single user (point to point) communication with the assistance of  $R$  parallel AF relay nodes.

### A. Point to Point Channel Model: $PTP(\mathbf{F}, P, \mathbf{G}, \{\mathbf{D}\}, P_R)$

The input-output equations for this case are as follows:

$$\mathbf{R} = \mathbf{F}x + \mathbf{N}_R, \quad (1)$$

$$y = \text{Tr}[\mathbf{G}\mathbf{D}(\mathbf{F}x + \mathbf{N}_R)] + n. \quad (2)$$

where  $\mathbf{R}$ ,  $\mathbf{F}$ ,  $\mathbf{G}$ ,  $\mathbf{D}$  and  $\mathbf{N}_R$  are  $R \times R$  diagonal matrices with the  $i^{\text{th}}$  principal diagonal terms  $r_i$ ,  $f_i$ ,  $g_i$ ,  $d_i$ ,  $n_{i,R}$  respectively representing the received signal at the  $i^{\text{th}}$  relay node, the channel coefficient from the source to the  $i^{\text{th}}$  relay node, the channel coefficient from the  $i^{\text{th}}$  relay node to the destination, the relay amplification factor for the  $i^{\text{th}}$  relay node, and the additive white Gaussian noise (AWGN) component at the  $i^{\text{th}}$  relay receiver, modeled as an i.i.d. zero mean unit variance Gaussian random variable.  $x$  is the symbol transmitted by the source,  $y$  is the symbol received by the destination and  $n$  is the zero mean unit variance AWGN at the destination node. The power constraints are specified as follows:

$$\text{Source Power Constraint:} \quad \mathbb{E}[x^2] = P, \quad (3)$$

$$\text{Relay Power Constraint:} \quad \mathbb{E}[\|\mathbf{D}(\mathbf{F}x + \mathbf{N}_R)\|^2] = \text{Tr}(\mathbf{D}^2(I + P\mathbf{F}^2)) = P_R. \quad (4)$$

The point to point channel under these assumptions is denoted as  $PTP(\mathbf{F}, P, \mathbf{G}, \{\mathbf{D}\}, P_R)$ . We use the notation  $\{\mathbf{D}\}$  to indicate all feasible relay amplification matrices while we use  $\mathbf{D}$  to indicate a specific choice of the relay amplification matrix. For example,  $PTP(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R)$  refers to the point to point channel with a given  $\mathbf{D}$  matrix. Similarly, while  $C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R)$  refers to the point to point channel capacity with a given relay amplification matrix  $\mathbf{D}$ ,  $C^{PTP}(\mathbf{F}, P, \mathbf{G}, \{\mathbf{D}\}, P_R)$  refers to the capacity optimized over all  $\mathbf{D}$  that satisfy the power constraints.

### B. Capacity and Optimal Relay Amplification

The capacity of the point to point parallel AF relay channel and the optimal relay amplification factors have been obtained previously by Maric and Yates [19]. We re-visit the result in order to introduce our notation and to illustrate the key ideas that are later applied to multiple access channels. Given any choice of relay amplification vector  $\mathbf{D}$  that satisfies the relay transmit power constraint, the resulting channel is an AWGN channel whose capacity is simply

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R) = \log \left( 1 + \frac{[\text{Tr}(\mathbf{G}\mathbf{D}\mathbf{F})]^2 P}{1 + \text{Tr}(\mathbf{D}^2\mathbf{G}^2)} \right). \quad (5)$$

Therefore, the capacity optimization problem for the point to point channel may be represented as:

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \{\mathbf{D}\}, P_R) = \max_{\mathbf{D} \in \mathcal{D}} C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R), \quad (6)$$

where  $\mathcal{D}$  is the set of feasible relay amplification matrices:

$$\mathcal{D} = \{ \mathbf{D} : \text{Tr}(\mathbf{D}^2(I + P\mathbf{F}^2)) = P_R \}.$$

The following result for the closed form capacity expression is presented in previous work by Larsson and Rong in [21], and Maric and Yates in [19].

**Theorem 1:** The capacity of the point to point channel described above is

$$\begin{aligned} C^{PTP}(P, \mathbf{F}, \{\mathbf{D}\}, P_R, \mathbf{G}) &= \log \left( 1 + PP_R \text{Tr} \left[ \mathbf{G}^2 \mathbf{F}^2 (I + P\mathbf{F}^2 + P_R \mathbf{G}^2)^{-1} \right] \right) \\ &= \log \left( 1 + PP_R \sum_{k=1}^R \frac{f_k^2 g_k^2}{1 + P f_k^2 + P_R g_k^2} \right). \end{aligned}$$

The optimum relay amplification vector for the point to point relay channel is given by

$$\mathbf{D} = \gamma \mathbf{F} \mathbf{G} (I + P\mathbf{F}^2 + P_R \mathbf{G}^2)^{-1}, \quad (7)$$

where  $\gamma$  is a constant necessary to satisfy the relay transmit power constraint, and may be expressed explicitly as

$$\gamma = \sqrt{\frac{P_R}{\text{Tr} \left( \mathbf{F}^2 \mathbf{G}^2 (I + P\mathbf{F}^2 + P_R \mathbf{G}^2)^{-2} (I + P\mathbf{F}^2) \right)}}. \quad (8)$$

### C. Reciprocity of the Point to Point AF Relay Channel

From the result of Theorem 1, notice that

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \{\mathbf{D}\}, P_R) = C^{PTP}(\mathbf{G}, P_R, \mathbf{F}, \{\mathbf{D}\}, P). \quad (9)$$

The capacity is unchanged if we switch variables as follows:

$$P \rightarrow P_R, \quad (10)$$

$$P_R \rightarrow P, \quad (11)$$

$$\mathbf{F} \rightarrow \mathbf{G}, \quad (12)$$

$$\mathbf{G} \rightarrow \mathbf{F}. \quad (13)$$

In other words, the capacity is the same if we switch the transmitter and the receiver as long as the powers of the transmitter and the relay are also switched. Fig. 1 shows the dual channels that have the same capacity.

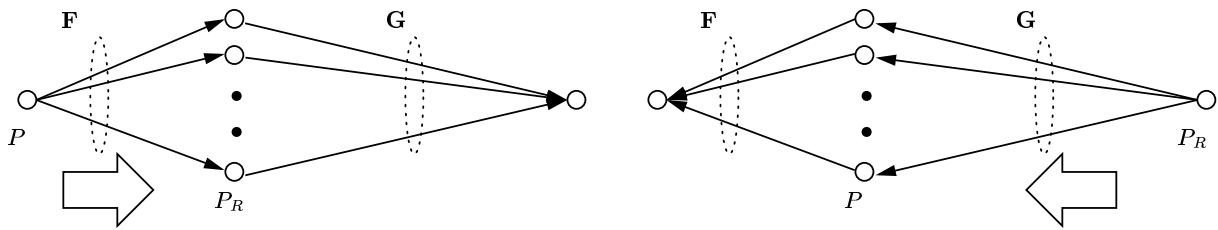


Fig. 1. Point to point dual channels

Further, note that the optimal relay amplification vector was found as

$$\mathbf{D} = \gamma \mathbf{F} \mathbf{G} (I + P \mathbf{F}^2 + P_R \mathbf{G}^2)^{-1}. \quad (14)$$

Note that except for the constant  $\gamma$  the optimal relay amplification vector is identical for the original channel and the dual channel, i.e., it is unaffected by the switch of variables in (10)-(13). However, the normalizing constant  $\gamma$  must be different on the two channels because the power constraints are different. For the original channel  $\gamma$  is given by Theorem 1, whereas on the dual channel we will have

$$\gamma_{\text{dual}} = \sqrt{\frac{P}{\text{Tr}(\mathbf{F}^2 \mathbf{G}^2 (I + P \mathbf{F}^2 + P_R \mathbf{G}^2)^{-2} (I + P_R \mathbf{G}^2))}}. \quad (15)$$

The following theorem shows that this duality property is actually much stronger as it holds not only for the optimal relay amplification matrix  $\mathbf{D}$  but rather for *every* feasible  $\mathbf{D}$ .

**Theorem 2:**

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R) = C^{PTP}(\mathbf{G}, P_R, \mathbf{F}, \kappa \mathbf{D}, P). \quad (16)$$

Given any relay amplification matrix  $\mathbf{D}$  for a point to point parallel AF relay channel  $PTP(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R)$  there exists a dual point to point parallel AF relay channel  $PTP(\mathbf{G}, P_R, \mathbf{F}, \kappa \mathbf{D}, P)$  that has the same capacity as  $PTP(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R)$  and where  $\kappa$  is chosen to satisfy the relay power constraint of the dual channel.

*Proof:* We start with the capacity and relay power constraint expressions of the original and the dual point to point channels.

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R) = \log \left( 1 + \frac{[\text{Tr}(\mathbf{G} \mathbf{D} \mathbf{F})]^2 P}{1 + \text{Tr}(\mathbf{D}^2 \mathbf{G}^2)} \right), \quad \text{Tr}(\mathbf{D}^2 (I + P \mathbf{F}^2)) = P_R. \quad (17)$$

$$C^{PTP}(\mathbf{G}, P_R, \mathbf{F}, \kappa \mathbf{D}, P) = \log \left( 1 + \frac{[\text{Tr}(\mathbf{F} \kappa \mathbf{D} \mathbf{G})]^2 P_R}{1 + \text{Tr}(\kappa^2 \mathbf{D}^2 \mathbf{F}^2)} \right), \quad \text{Tr}(\kappa^2 \mathbf{D}^2 (I + P_R \mathbf{G}^2)) = P. \quad (18)$$

Substituting from the power constraint into the capacity expression we have:

$$C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R) = \log \left( 1 + \frac{[\text{Tr}(\mathbf{G} \mathbf{D} \mathbf{F})]^2 P}{\frac{\text{Tr}(\mathbf{D}^2 (I + P \mathbf{F}^2))}{P_R} + \text{Tr}(\mathbf{D}^2 \mathbf{G}^2)} \right) \quad (19)$$

$$= \log \left( 1 + \frac{[\text{Tr}(\mathbf{G} \mathbf{D} \mathbf{F})]^2 P P_R}{\text{Tr}(\mathbf{D}^2 (I + P \mathbf{F}^2 + P_R \mathbf{G}^2))} \right). \quad (20)$$

$$C^{PTP}(\mathbf{G}, P_R, \mathbf{F}, \kappa \mathbf{D}, P) = \log \left( 1 + \frac{[\text{Tr}(\mathbf{F} \kappa \mathbf{D} \mathbf{G})]^2 P_R}{\frac{\text{Tr}(\kappa^2 \mathbf{D}^2 (I + P_R \mathbf{G}^2))}{P} + \text{Tr}(\kappa^2 \mathbf{D}^2 \mathbf{F}^2)} \right) \quad (21)$$

$$= \log \left( 1 + \frac{[\text{Tr}(\mathbf{G} \mathbf{D} \mathbf{F})]^2 P P_R}{\text{Tr}(\mathbf{D}^2 (I + P \mathbf{F}^2 + P_R \mathbf{G}^2))} \right). \quad (22)$$

Thus,  $C^{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R) = C^{PTP}(\mathbf{G}, P_R, \mathbf{F}, \kappa \mathbf{D}, P)$ . ■

The proof of Theorem 2 illustrates an important point. By normalizing the noise power to unity and also incorporating the relay power constraint into the normalized channel definition  $\text{PTP}(\mathbf{F}, P, \mathbf{G}, \mathbf{D}, P_R)$  may be represented as:

$$y = \frac{\text{Tr}(\mathbf{GDF})\sqrt{P_R}}{\sqrt{\text{Tr}(\mathbf{D}^2(I + P\mathbf{F}^2 + P_R\mathbf{G}^2))}}x + n. \quad (23)$$

Comparing this to the normalized representation of the dual  $\text{PTP}(\mathbf{G}, P_R, \mathbf{F}, \kappa\mathbf{D}, P)$ , it is clear that the normalized dual channel is identical to the original channel.

The appealing feature of this normalized form is that it is unaffected by any scaling of the relay amplification matrix  $\mathbf{D}$  by any constant  $\kappa$ . While there is only a unique scaling factor for any diagonal matrix such that it will satisfy the relay transmit power constraint, that scaling factor does not affect the normalized channel as the power constraint is already accommodated into this form. Further, the reciprocity of the point to point channel is also evident from the normalized form.

#### IV. PARALLEL AF RELAY MAC AND BC

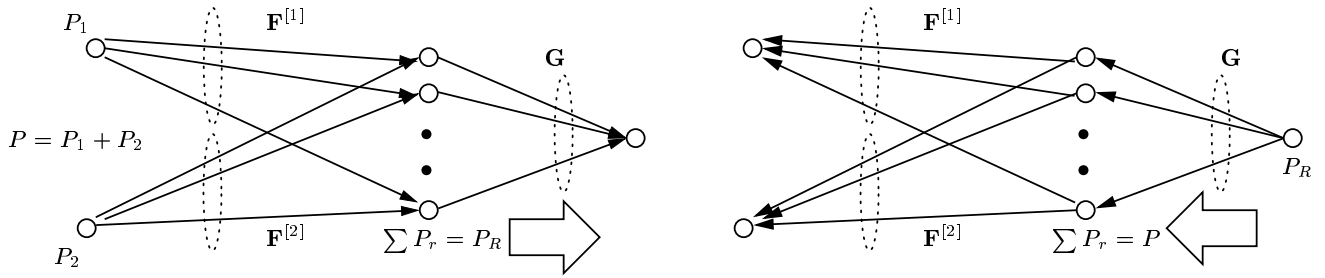


Fig. 2. Dual Parallel AF Relay MAC and BC channels

For simplicity we focus on the two user case. The AF relay MAC and its dual BC models are depicted in Fig. 2.

### A. Symbol Definitions and Notation

Variable	AF MAC	Dual AF BC
$R$	Number of parallel AF relays	Number of parallel AF relays
$P_1$	Transmit power for source 1	
$P_2$	Transmit power for source 2	
$P$	$P_1 + P_2$	Total transmit power of $R$ relays
$P_R$	Total transmit power of $R$ relays	Source transmit power
$f_i^{[j]}$	Channel from $j^{th}$ source to $i^{th}$ relay	Channel from $i^{th}$ relay to $j^{th}$ destination.
$g_i$	Channel from $i^{th}$ relay to destination	Channel from source to $i^{th}$ relay.
$d_i$	Amplification factor at the $i^{th}$ relay	Amplification factor at the $i^{th}$ relay
$x$		Transmitted symbol for common source
$x_i$	Transmitted symbol for source $i$	
$y$	Received symbol at common destination	
$y_i$		Received symbol at $i^{th}$ destination
$r_i$	Received symbol at the $i^{th}$ relay	Received symbol at the $i^{th}$ relay
$(d_i)(r_i)$	Symbol transmitted by the $i^{th}$ relay	Symbol transmitted by the $i^{th}$ relay
$n_{R,i}$	Unit power AWGN at the $i^{th}$ relay	Unit power AWGN at the $i^{th}$ relay
$n_i$		Unit power AWGN at $i^{th}$ destination
$n$	Unit power AWGN at common destination	

### B. Two user Parallel AF Relay MAC and BC Models

We use the notation  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$  to denote the two user multiple access channel described above, i.e. with the channels between the transmitters and relays  $\mathbf{F}^{[1]}, \mathbf{F}^{[2]}$ , the corresponding source transmit powers  $P_1, P_2$  (respectively), the channel between the relays and the destination  $\mathbf{G}$ , the relay amplification vector  $\mathbf{D}$ , and total transmit power at all relays  $P_R$ . To distinguish the MAC resulting from a specific choice of  $\mathbf{D}$  from the MAC where all  $\mathbf{D}$  from the feasible set are allowed, we denote the former as  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$  and the latter as  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ .

1)  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ : For the two user parallel AF relay  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ , the received signals at the relays and the common destination are as follows:

$$\mathbf{R} = \mathbf{F}^{[1]}x_1 + \mathbf{F}^{[2]}x_2 + \mathbf{N}_R, \quad (24)$$

$$y = \text{Tr}[\mathbf{GD} (\mathbf{F}^{[1]}x_1 + \mathbf{F}^{[2]}x_2 + \mathbf{N}_R)] + n. \quad (25)$$

Normalizing the overall AWGN (including the noise forwarded by the relay nodes) to unit variance, the destination output can be expressed as

$$y' = \frac{\text{Tr}(\mathbf{GDF}^{[1]})}{\sqrt{1 + \text{Tr}(\mathbf{D}^2 \mathbf{G}^2)}} x_1 + \frac{\text{Tr}(\mathbf{GDF}^{[2]})}{\sqrt{1 + \text{Tr}(\mathbf{D}^2 \mathbf{G}^2)}} x_2 + n'. \quad (26)$$

The power constraints are:

$$\mathbb{E}[x_1^2] = P_1, \quad (27)$$

$$\mathbb{E}[x_2^2] = P_2, \quad (28)$$

$$\text{Tr}\left(\mathbf{D}^2 \left(I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2}\right)\right) = P_R. \quad (29)$$

Note that the power constraints are specified as equalities rather than inequalities. The equality constraint is important for the results of Theorem 3, Theorem 4, and Theorem 5 that appear in Section V. The significance of the equality assumption for the power constraints is explained in the discussion at the end of Section V-B. As in the point to point case, the power constraint can be substituted into the channel output equation to obtain a corresponding normalized form

$$y' = \frac{\text{Tr}(\mathbf{GDF}^{[1]}) \sqrt{P_R}}{\sqrt{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]}} x_1 + \frac{\text{Tr}(\mathbf{GDF}^{[2]}) \sqrt{P_R}}{\sqrt{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]}} x_2 + n'. \quad (30)$$

2)  $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \{\mathbf{D}\}, P)$ : We use the shorthand notation  $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \mathbf{D}, P)$  to indicate the dual BC, i.e. the broadcast channel with transmit power  $P_R$ , channel vector  $\mathbf{G}$  from transmitter to relays, channel vectors  $\mathbf{F}^{[1]}$  and  $\mathbf{F}^{[2]}$  from the relays to receiver 1 and 2 respectively, relay amplification factor given by  $\mathbf{D}$  and total transmit power used by the relays  $P$ . As for the MAC, we use  $\{\mathbf{D}\}$  to indicate all feasible relay amplification factors are allowed and  $\mathbf{D}$  to indicate a specific choice. For the dual broadcast channel, received signals at the relays and the two destinations are as follows:

$$\mathbf{R} = \mathbf{G}x + \mathbf{N}_R, \quad (31)$$

$$y_1 = \text{Tr}[\mathbf{F}^{[1]}\mathbf{D}(\mathbf{G}x + \mathbf{N}_R)] + n_1, \quad (32)$$

$$y_2 = \text{Tr}[\mathbf{F}^{[2]}\mathbf{D}(\mathbf{G}x + \mathbf{N}_R)] + n_2, \quad (33)$$

with power constraints

$$\mathbb{E}[x^2] = P_R, \quad (34)$$

$$\text{Tr}(\mathbf{D}^2 (I + P_R \mathbf{G}^2)) = P. \quad (35)$$

Notice that the relays are associated with power  $P$  and the source with power  $P_R$ . Normalizing the overall AWGN (including the noise forwarded by the relays) to unit power,

$$y'_1 = \frac{\text{Tr}(\mathbf{F}^{[1]}\mathbf{D}\mathbf{G})}{\sqrt{1 + \text{Tr}(\mathbf{D}^2 \mathbf{F}^{[1]2})}} x + n'_1, \quad (36)$$

$$y_2' = \frac{\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G})}{\sqrt{1 + \text{Tr}(\mathbf{D}^2\mathbf{F}^{[2]2})}} x + n_2'. \quad (37)$$

Incorporating the relay power constraint into the normalized channel,

$$y_1' = \frac{\text{Tr}(\mathbf{F}^{[1]}\mathbf{D}\mathbf{G}) \sqrt{P}}{\sqrt{\text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[1]2} + P_R\mathbf{G}^2)]}} x + n_1', \quad (38)$$

$$y_2' = \frac{\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G}) \sqrt{P}}{\sqrt{\text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)]}} x + n_2'. \quad (39)$$

Recall that for the point to point case, the dual channel and the original channel were identical. Even for the conventional MAC-BC duality, the channels on the MAC and BC are identical. However, the forms of the equivalent normalized channels for the MAC and BC presented above are significantly different. In particular, while for the BC, user 1's effective channel gain in (38) is independent of  $\mathbf{F}^{[2]}$  and user 2's effective channel gain in (39) is independent of  $\mathbf{F}^{[1]}$ , for the MAC the effective channel gain of user 1 (as well as user 2) in (30) depends on both  $\mathbf{F}^{[1]}$ ,  $\mathbf{F}^{[2]}$ . A duality relationship between these two channels is therefore not a trivial observation based on their normalized channels. What makes the existence of a duality relationship even more unlikely is the fact that the channels themselves depend on how the total power is split among the users in the multiple access channel. With these apparent complexities, it is rather surprising that a precise duality relationship does exist between the MAC and BC described above, as we show in Section VI-A.

## V. RELAY OPTIMIZATION FOR PARALLEL AF RELAY MAC

Given a relay amplification vector  $\mathbf{D}$  the capacity region of the resulting scalar Gaussian MAC is the well known pentagon that can be expressed as:

$$\mathcal{C}^{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R) = \{(R_1, R_2) \quad : \quad \begin{aligned} R_1 &\leq \log\left(1 + P_1 \frac{[\text{Tr}(\mathbf{D}\mathbf{G}\mathbf{F}^{[1]})]^2}{1 + \text{Tr}(\mathbf{D}^2\mathbf{G}^2)}\right), \\ R_2 &\leq \log\left(1 + P_2 \frac{[\text{Tr}(\mathbf{D}\mathbf{G}\mathbf{F}^{[2]})]^2}{1 + \text{Tr}(\mathbf{D}^2\mathbf{G}^2)}\right), \\ R_1 + R_2 &\leq \left(1 + \frac{P_1 [\text{Tr}(\mathbf{D}\mathbf{G}\mathbf{F}^{[1]})]^2 + P_2 [\text{Tr}(\mathbf{D}\mathbf{G}\mathbf{F}^{[2]})]^2}{1 + \text{Tr}(\mathbf{D}^2\mathbf{G}^2)}\right) \}. \end{aligned}$$

Taking the union over all  $\mathbf{D}$  that satisfy the relay sum power constraint gives us a characterization of the rate region of  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$  as:

$$\mathcal{C}^{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R) = \cup_{\mathbf{D} \in \mathcal{D}_{MAC}} \mathcal{C}^{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R), \quad (40)$$

where

$$\mathcal{D}_{MAC} = \left\{ \mathbf{D} : \text{Tr}\left(\mathbf{D}^2\left(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2}\right)\right) = P_R \right\} \quad (41)$$

is the set of all relay amplification factors that satisfy the total transmit power constraint at the relays.<sup>12</sup> Note that in the absence of any further characterization of  $\mathbf{D}$  we are left with optimization over the entire space of feasible  $\mathbf{D}$ , i.e. a  $R$  dimensional space. A brute force solution to such an optimization may be difficult as the number of relays increases. Theorem 3 solves this problem.

### A. Relay Optimization

The following theorem reveals the structure of the optimal relay amplification factor  $\mathbf{D}$  for rate pairs on the boundary of the rate region.

**Theorem 3:** The optimal relay amplification matrix  $\mathbf{D}$  to maximize any weighted sum of users' rates  $\mu_1 R_1 + \mu_2 R_2$  with  $\mu_1, \mu_2 \geq 0$ , has the following form:

$$\mathbf{D}(\theta) = \gamma \mathbf{G} \left( P_1 \mathbf{F}^{[1]} \sin \theta + P_2 \mathbf{F}^{[2]} \cos \theta \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \quad (42)$$

and  $\gamma$  may be expressed explicitly as:

$$\gamma = \sqrt{\frac{P_R}{\text{Tr} \left( \mathbf{G}^2 \left( P_1 \mathbf{F}^{[1]} \sin \theta + P_2 \mathbf{F}^{[2]} \cos \theta \right)^2 \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-2} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} \right) \right)}}.$$

The proof of Theorem 3 is presented in Appendix I. Note that the optimization space over  $\mathbf{D}$  is now only one dimensional, as opposed to the original  $R$  dimensional space. To identify a point on the boundary of the rate region one only needs the corresponding  $\theta$ . Therefore, the angle  $\theta$  in Theorem 3 plays an important role. As we will establish in the following Theorems,  $|\theta|$  going from 0 to  $\pi/2$  describes the boundary of the rate region, with  $\theta = 0$  corresponding to the point where user 2 achieves his maximum rate,  $\theta^{11}$  (defined in Theorem 5) corresponds to the points where the sum rate is maximized, and  $|\theta| = \pi/2$  corresponds to the point where user 1 achieves his maximum rate. This is also depicted in Fig 3.

Next we characterize the rate region explicitly by obtaining in closed form the rate pairs corresponding to the maximum sum rate as well as the maximum individual user rates. We will also present a system of equations whose solution is the rate pair  $(R_1, R_2)$  that maximizes  $\mu_1 R_1 + \mu_2 R_2$  for any non-negative  $\mu_1, \mu_2$ . We start with the extreme point  $\mu_1 = 1, \mu_2 = 0$  that characterizes the maximum possible rate of user 1.

### B. Maximum Individual Rate $R_1$

**Theorem 4:** For the MAC  $(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ , the maximum rate  $R_1$  that can be supported is

$$C_1^{10} = \log \left( 1 + P_1 P_R \text{Tr} \left( \mathbf{G}^2 \mathbf{F}^{[1]2} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \right) \right) \quad (43)$$

$$= \log \left( 1 + P_1 P_R \sum_{k=1}^R \frac{f_k^{[1]2} g_k^2}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2} \right). \quad (44)$$

This rate is achieved with relay amplification matrix  $\mathbf{D}(\theta = \pi/2)$ ,

$$\mathbf{D}(\pi/2) = \gamma P_1 \mathbf{G} \mathbf{F}^{[1]} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1}. \quad (45)$$

Further, if user 1 achieves his maximum rate  $C_1^{10}$ , i.e.  $\mathbf{D} = \mathbf{D}(\pi/2)$ , then the maximum rate that can be achieved by user 2 is

$$C_2^{10} = \log \left( 1 + P_2 P_R \frac{\left( \sum_{i=1}^R \frac{g_i^2 f_i^{[1]} f_i^{[2]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \right)^2}{\sum_{i=1}^R \frac{g_i^2 f_i^{[1]2}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} + P_1 P_R \left( \sum_{i=1}^R \frac{g_i^2 f_i^{[1]2}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \right)^2} \right).$$

The proof of Theorem 4 is in Appendix II.

By symmetry, the expression for maximum rate  $C_2^{01}$  for user 2 is obtained by switching indices 1 and 2 in Theorem 4. Note that user 2 obtains his maximum rate when  $\theta = 0$ , i.e., with amplification matrix  $\mathbf{D}(0)$ .

Theorem 4 leads to an interesting observation. Notice that  $C_1^{10}$ , i.e., the maximum rate for user 1 in  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$  is not the same as the maximum rate of user 1 if user 2 was *not* transmitting. In fact, for  $P_1, P_2 > 0$ ,

$$C_1^{10} < C^{PTP}(\mathbf{F}^{[1]}, P_1, \mathbf{G}, \{\mathbf{D}\}, P_R). \quad (46)$$

Recall that in the conventional Gaussian MAC (i.e. without AF relays) the maximum rate that user 1 can achieve is the same as his channel capacity *as if* user 2 is not transmitting. In the conventional Gaussian MAC, even though user 2 transmits power  $P_2$ , user 1 can achieve his single user capacity with power  $P_1$  if the rate  $R_2$  allows user 2 to be decoded and subtracted out before user 1 is decoded. The reason this is not true in the relay MAC is because of the power constraint at the relays. Even though user 2 can be decoded and his signal subtracted out at the destination, it does not make the system equivalent to one where user 2 was not transmitting at all. Transmission by user 2 affects the power constraint at the AF relays, thereby strictly reducing user 1's maximum rate as compared to the case when user 2 is silent.

Next we consider the sum rate, i.e.  $\mu_1 = \mu_2 = 1$ .

### C. Maximum Sum Rate

**Theorem 5:** The maximum sum rate  $C^{11}$  of  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ , (i.e., a 2 user MAC with  $R$  parallel AF relays, where the user's transmit powers are  $P_1$  and  $P_2$ , the total transmit power of all relays is  $P_R$  and the relay amplification factor is optimized over all feasible  $\mathbf{D}$ ) is given by:

$$\begin{aligned} C^{11} &= \log(1 + \text{SNR}^*), \\ \text{SNR}^* &= \frac{P_R}{2} \left( P_1 A_{11} + P_2 A_{22} + \sqrt{(P_1 A_{11} + P_2 A_{22})^2 - 4P_1 P_2 (A_{11} A_{22} - A_{12}^2)} \right), \end{aligned}$$

where

$$\begin{aligned}
A_{11} &= \sum_{k=1}^R \frac{g_k^2 f_k^{[1]2}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2} = \text{Tr} \left( \mathbf{G}^2 \mathbf{F}^{[1]2} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \right), \\
A_{22} &= \sum_{k=1}^R \frac{g_k^2 f_k^{[2]2}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2} = \text{Tr} \left( \mathbf{G}^2 \mathbf{F}^{[2]2} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \right), \\
A_{12} &= \sum_{k=1}^R \frac{g_k^2 f_k^{[1]} f_k^{[2]}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2} = \text{Tr} \left( \mathbf{G}^2 \mathbf{F}^{[1]} \mathbf{F}^{[2]} \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \right).
\end{aligned}$$

The maximum sum rate is achieved with relay amplification matrix  $\mathbf{D}(\theta = \theta^{11})$ , where

$$\theta^{11} = \tan^{-1} \left( \frac{P_R P_2 A_{12}}{\text{SNR}^* - P_R P_1 A_{11}} \right). \quad (47)$$

Further, the following rates pairs  $(R_1, R_2)$  are sum rate optimal:

$$\begin{aligned}
(R_1^{11}(2 \rightarrow 1), R_2^{11}(2 \rightarrow 1)) &= \left( \log(1 + \beta \text{SNR}^*), \log \left( 1 + \frac{(1 - \beta) \text{SNR}^*}{1 + \beta \text{SNR}^*} \right) \right), \\
(R_1^{11}(1 \rightarrow 2), R_2^{11}(1 \rightarrow 2)) &= \left( \log \left( 1 + \frac{\beta \text{SNR}^*}{1 + (1 - \beta) \text{SNR}^*} \right), \log(1 + (1 - \beta) \text{SNR}^*) \right), \\
\text{where } \beta &= \frac{\text{SNR}^* - P_2 P_R A_{22}}{2 \text{SNR}^* - P_1 P_R A_{11} - P_2 P_R A_{22}}.
\end{aligned}$$

The rate pair  $(R_1^{11}(2 \rightarrow 1), R_2^{11}(2 \rightarrow 1))$  is achieved by successive decoding of user 2 followed by user 1, while  $(R_1^{11}(1 \rightarrow 2), R_2^{11}(1 \rightarrow 2))$  is achieved by successive decoding of user 1 followed by user 2.

Proof of Theorem 5 is presented in Appendix III. Notice from Theorem 5 that the sum rate maximizing point is not unique.

#### D. Optimal Rate Pair to maximize $\mu_1 R_1 + \mu_2 R_2$

**Theorem 6:** For any  $\mu_1, \mu_2$  with  $\mu_1 \geq \mu_2 \geq 0$ , the weighted sum rate  $\mu_1 R_1 + \mu_2 R_2$  for the MAC( $\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R$ ) (i.e., a 2 user MAC with  $R$  parallel AF relays, where the user's transmit powers are equal to  $P_1$  and  $P_2$ , the total transmit power of all relays is  $P_R$  and the relay amplification factor is optimized over all feasible  $\mathbf{D}$ ), is maximized by the rate pair  $(R_1^{\mu_1, \mu_2}, R_2^{\mu_1, \mu_2})$  given by

$$R_1^{\mu_1, \mu_2} = \log(1 + \text{SNR}_1^*), \quad R_2^{\mu_1, \mu_2} = \log \left( 1 + \frac{\text{SNR}_2^*}{1 + \text{SNR}_1^*} \right), \quad (48)$$

where  $\text{SNR}_1^*$  and  $\text{SNR}_2^*$  are the solutions to the following simultaneous equations:

$$\mu_1 = \frac{\mu_2}{\text{SNR}_1^* + \text{SNR}_2^*} (1 + P_R P_1 A_{11} + P_R P_2 A_{12} / \alpha) + \frac{\mu_1 - \mu_2}{1 + \text{SNR}_1^*} (1 + P_R P_1 A_{11}), \quad (49)$$

$$\mu_1 = \frac{\mu_2}{\text{SNR}_1^* + \text{SNR}_2^*} (1 + P_R P_1 A_{12} \alpha + P_R P_2 A_{22}) + \frac{\mu_1 - \mu_2}{1 + \text{SNR}_1^*} (1 + P_R P_1 A_{12} \alpha), \quad (50)$$

$$\alpha = \sqrt{\frac{P_1 \text{SNR}_1^*}{P_2 \text{SNR}_2^*}}. \quad (51)$$

Proof of Theorem 6 is provided in Appendix IV. Note that for  $\mu_2 \geq \mu_1$  the corresponding rate pairs can be obtained by switching indices 1, 2 in Theorem 6. While the simultaneous equations do not appear to allow a closed form solution for general  $\mu_1, \mu_2$ , closed form solutions can be obtained for the special cases of  $\mu_1 = 1, \mu_2 = 0$  and  $\mu_1 = \mu_2 = 1$  as discussed above. In general, Theorem 6 allows a relatively straightforward numerical solution that is much easier than a brute force optimization over all feasible  $\mathbf{D}$ . Solving the weighted sum of rate pairs optimization problem is especially useful for rate allocation problems as well as to plot the entire rate region of the parallel AF relay MAC.

*E. The Parallel AF Relay MAC Rate Region*

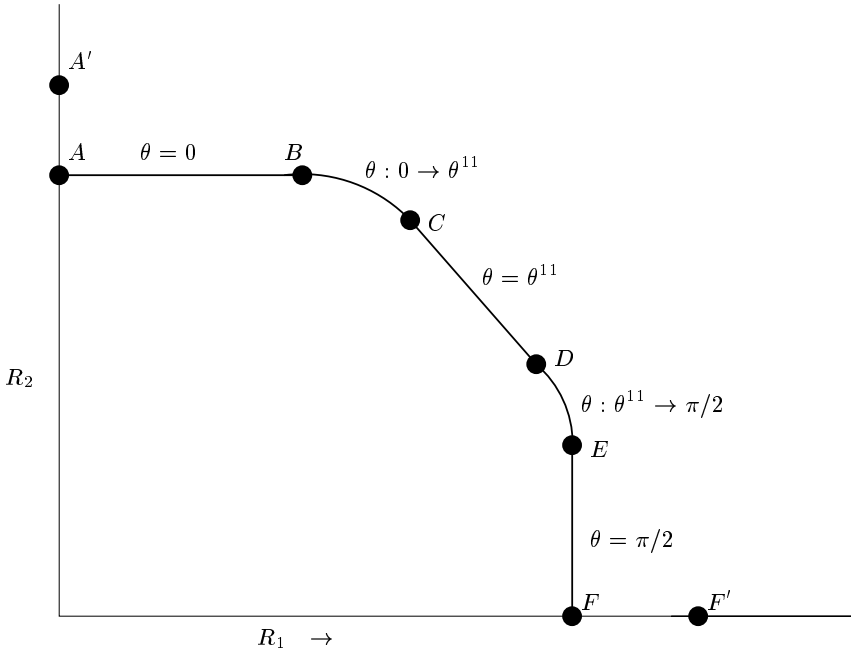


Fig. 3. Rate region of two user AF relay MAC  $(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$  with user 1 and 2's transmit powers equal to  $P_1, P_2$ , respectively.

Combining the results of Theorems 3,4,5 and 6 the boundary of the rate region for the parallel AF relay MAC  $(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$  is graphically plotted by the following algorithm. The steps are labeled with reference to Fig. 3.

- A → B: Draw a straight horizontal line from  $(R_1, R_2) = (0, C_2^{01})$  to  $(R_1, R_2) = (C_1^{01}, C_2^{01})$ .

- B → C: For  $\theta : 0 \rightarrow \theta^{11}$

$$\begin{aligned} \mathbf{D}(\theta) &= \mathbf{G} \left( P_1 \mathbf{F}^{[1]} \sin \theta + P_2 \mathbf{F}^{[2]} \cos \theta \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \\ R_1 &= \log \left( 1 + P_1 P_R \frac{[\text{Tr}(\mathbf{GDF}^{[1]})]^2}{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]} + P_2 P_R [\text{Tr}(\mathbf{GDF}^{[2]})]^2 \right) \\ R_2 &= \log \left( 1 + P_2 P_R \frac{[\text{Tr}(\mathbf{GDF}^{[2]})]^2}{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]} \right) \end{aligned} \quad (52)$$

- C → D: Draw a straight line from  $(R_1^{11}(1 \rightarrow 2), R_2^{11}(1 \rightarrow 2))$  to  $(R_1^{11}(2 \rightarrow 1), R_2^{11}(2 \rightarrow 1))$ .
- D → E: For  $\theta : \theta^{11} \rightarrow \text{sgn}(\theta^{11})\frac{\pi}{2}$

$$\begin{aligned} \mathbf{D}(\theta) &= \mathbf{G} \left( P_1 \mathbf{F}^{[1]} \sin \theta + P_2 \mathbf{F}^{[2]} \cos \theta \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \\ R_1 &= \log \left( 1 + P_1 P_R \frac{[\text{Tr}(\mathbf{GDF}^{[1]})]^2}{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]} \right) \\ R_2 &= \log \left( 1 + P_2 P_R \frac{[\text{Tr}(\mathbf{GDF}^{[2]})]^2}{\text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]} + P_1 P_R [\text{Tr}(\mathbf{GDF}^{[1]})]^2 \right) \end{aligned} \quad (53)$$

- E → F: Draw a straight vertical line from  $(R_1, R_2) = (C_1^{10}, C_2^{10})$  to  $(R_1, R_2) = (C_1^{10}, 0)$ .

Fig. 3 shows the typical shape of the rate region. Optimal relay amplification factor is indicated on the figure in terms of the parameter  $|\theta|$ .  $\theta$  is equal to zero between points *A* and *B*, it changes from 0 to  $\theta^{11}$  (as defined in Theorem 5) in the curved portion from points *B* to *C*.  $\theta$  is constant at  $\theta^{11}$  between points *C* to *D*.  $\theta$  changes from  $\theta^{11}$  to  $\text{sgn}(\theta^{11})\pi/2$  as we traverse the boundary from *E*, and it is again constant at  $\theta = \pi/2$  from the point *E* to point *F*. The coordinates of the points *A*, *B*, *C*, *D*, *E*, *F* are all known in closed form as given by the preceding results in this section. Point *A'* outside the rate region indicates the maximum rate of user 2 with power  $P_2$  if user 1 is not transmitting, i.e.  $P_1 = 0$ . This is strictly higher than point *A* which corresponds to the maximum rate of user 2 with power  $P_2$  when user 1 is sending constant symbols ( $R_1 = 0$ ) that can be subtracted out at the receiver. As explained earlier, unlike the conventional Gaussian MAC, user 1 sending constant symbols with power  $P_1$  is not identical to user 1 silent, even though the constant symbols can be subtracted by the destination. This is because transmission by user 1 affects the relay power constraint. Notice that AF relays are only allowed to scale the input, i.e. they cannot subtract the constant signal from user 1. Similarly, point *F'* indicates the maximum rate of user 1 if user 2 is not transmitting, which is higher than the maximum rate of user 1 when user 2 is transmitting (point *F*).

## VI. DUALITY RELATIONSHIPS IN PARALLEL AF RELAY NETWORKS

In this section we examine the duality of the parallel AF relay MAC and BC. In the previous section we obtained the rate region of the parallel AF relay MAC. Knowing the rate region of the AF relay MAC would enable us to find the rate region of the AF relay BC if we could obtain a duality relationship between the two. Establishing such a duality is the goal of this section.

### A. Duality of Parallel AF Relay MAC and BC

The following two theorems establish the duality relationship between the parallel AF relay MAC and BC.

**Theorem 7:**

$$\mathcal{C}^{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R) \subset \mathcal{C}^{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2) \quad (54)$$

Given any relay amplification matrix  $\mathbf{D}$  that satisfies the power constraint on the parallel AF relay multiple access channel  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$ , there exists a dual parallel AF relay broadcast channel  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$  such that any rate pair  $(R_1, R_2)$  that can be achieved on  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$  can also be achieved on  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\{\mathbf{D}\}, P)$ .  $\kappa$  is chosen to satisfy the relay sum power constraint on  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$ .

**Theorem 8:**

$$\mathcal{C}^{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \mathbf{D}, P) = \cup_{P_1, P_2 \geq 0, P_1 + P_2 = P} \mathcal{C}^{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \kappa\mathbf{D}, P_R) \quad (55)$$

Given any relay amplification matrix  $\mathbf{D}$  that satisfies the power constraint on the parallel AF relay broadcast channel  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \mathbf{D}, P)$  and given a rate pair  $(R_1, R_2)$  that is achievable on this parallel AF relay broadcast channel, there exist  $P_1, P_2 \geq 0$  such that  $P_1 + P_2 = P$  and a dual multiple access channel  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \kappa\mathbf{D}, P_R)$  such that the rate pair  $R_1, R_2$  is achievable on  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \kappa\mathbf{D}, P_R)$ .  $\kappa$  is chosen to satisfy the relay sum power constraint on  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$ .

The proof of Theorems 7 and 8 is presented in Appendix V.

Note that the duality relationship is a strong duality in the sense that the parallel AF relay MAC and BC are duals not only for the optimal relay amplification matrix  $\mathbf{D}$  but also for *any* feasible  $\mathbf{D}$ . This is an equally strong result as for the point to point case. Moreover, as in the point to point case, note that the powers of the relays and the transmitter are switched in the dual channel.

### B. Rate Region of the Parallel AF Relay BC

The duality relationship described in the previous section provides a method to compute the rate region of the parallel AF broadcast channel. As shown in Fig. 4 the BC rate region may be found simply as the union of the MAC rate regions.

### C. AF Relay MAC-BC Duality with Multiple Antenna Relays and Multiple Hops

The AF relay MAC-BC duality shown in section VI-A is not limited to single antenna relays or two hop networks with parallel relays. The same duality relationship holds when some of the relays have multiple antennas. It holds when signals and channel coefficients are complex. The duality also holds when more than 2 hops are considered. Thus, the MAC-BC duality holds for AF relay networks whether they are

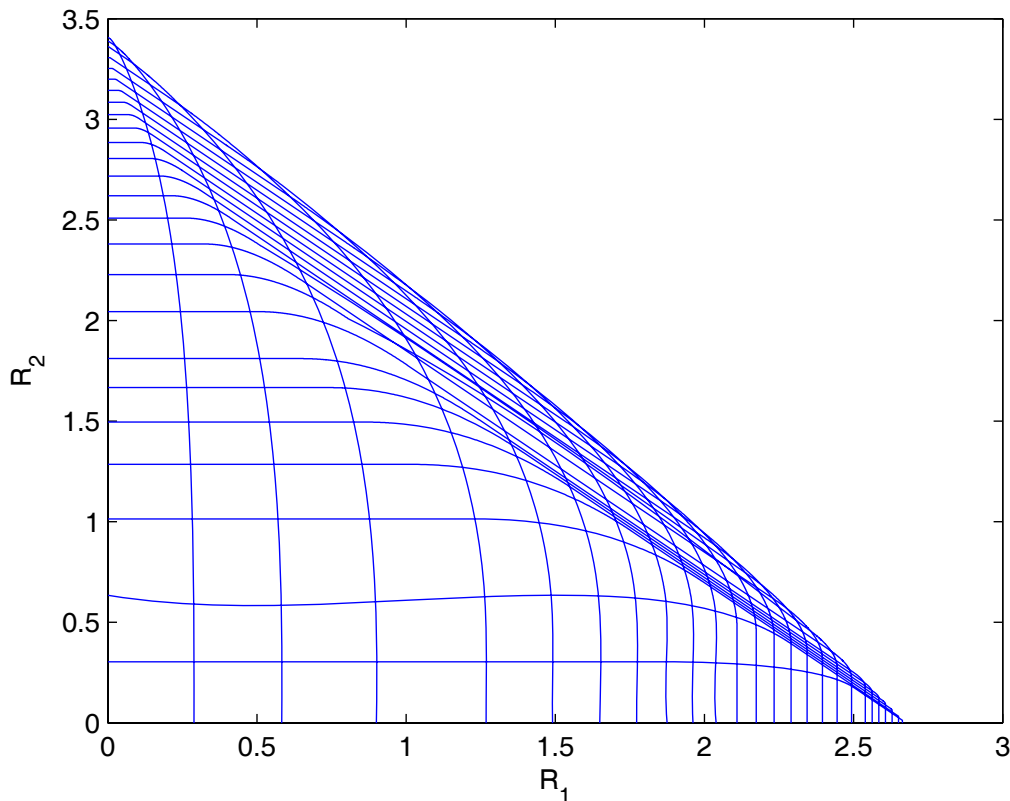


Fig. 4. AF relay BC rate region as the union of AF relay MAC rate regions

purely parallel (two hop), purely serial (multiple hops with a single relay at each hop), or several parallel AF relay clusters connected in series to form a multihop relay network. It holds whether the relays are distributed with a single antenna at each relay or they are able to cooperate as a multiple antenna node. The dual network in general is defined as the network where the MAC source nodes become the BC destination nodes and the MAC destination node becomes the BC source node. The power constraints are changed so that the channel on each hop is subject to the same transmit power as in the dual network. In this section we prove the duality relationship in the general case with complex signals and channel coefficients, multiple hops and multiple antennas at the relays. However we still assume that the source and destination nodes are single antenna nodes.

Consider a  $Q + 1$  hop AF relay MAC as shown in Fig. 5. The hops are indexed by the variable  $q$  that takes values  $1, 2, \dots, Q + 1$ . All channels and signals are complex. The AF relays can have multiple antennas. The number of relays at each hop may be different and the relays at each hop may have a different number of antennas. The total number of transmitting relays at the  $q^{th}$  hop is  $r_{q-1}$  and the total number of receiving relays is  $r_q$ . The number of antennas at the  $i^{th}$  transmitting relay

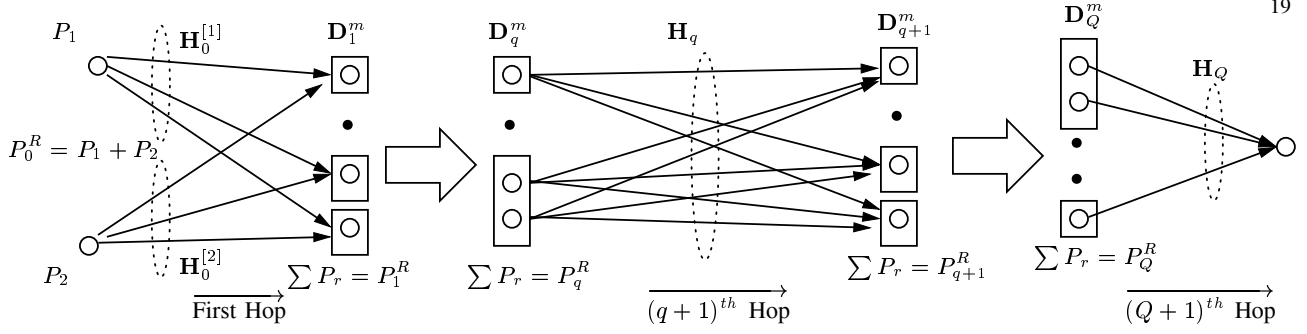


Fig. 5.  $(Q + 1)$ -hop MAC with distributed multiple antenna AF relays

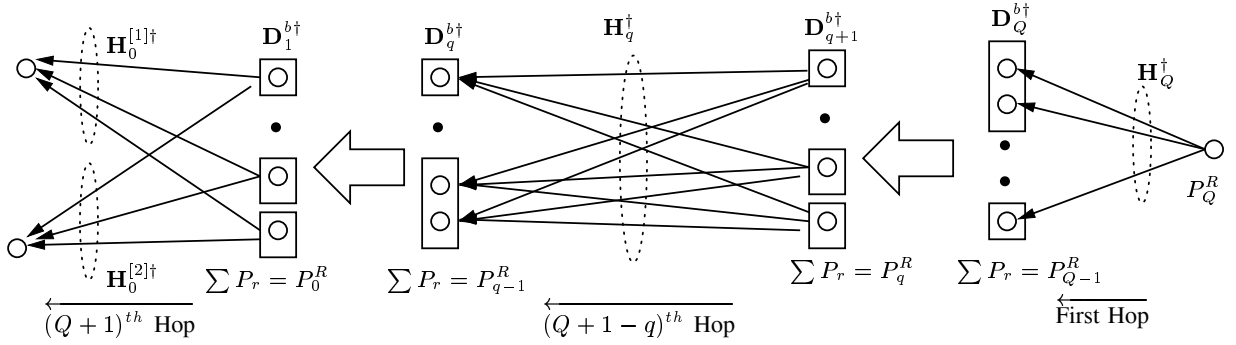


Fig. 6. Dual  $(Q + 1)$ -hop BC with distributed multiple antenna AF relays

on the  $q^{th}$  hop is  $a(i, q - 1)$ . The total number of transmitting antennas at the  $q^{th}$  hop is defined as  $R_{q-1} = a(1, q - 1) + a(2, q - 1) + \dots + a(r_{q-1}, q - 1)$ . Multiple antenna AF relays can forward any linear transformation of their inputs. In other words, the amplification factor for the  $i^{th}$  transmitting relay on the  $q^{th}$  hop is an  $a(i, q - 1) \times a(i, q - 1)$  matrix that we denote as  $\mathbf{D}^m(i, q - 1)$ . The overall amplification factor of the transmitting relays in the  $q^{th}$  hop of the relay network can be represented then as a block diagonal matrix  $\mathbf{D}_{q-1}^m = \text{BlockDiag}[\mathbf{D}^m(1, q - 1), \mathbf{D}^m(2, q - 1), \dots, \mathbf{D}^m(r_{q-1}, q - 1)]$  with overall dimensions  $R_{q-1} \times R_{q-1}$ . The superscript  $m$  indicates the MAC. The channel matrices are numbered according to the hops in the MAC. Thus,  $\mathbf{H}_{q-1}$  represents the  $R_q \times R_{q-1}$  matrix for the channel experienced by the signal on the  $q^{th}$  hop. Notice that this channel has a total of  $R_q$  receive antennas and a total of  $R_{q-1}$  transmit antennas. We consider only single antenna source and destination. Therefore, we define  $R_0 = 2$  (two single antenna sources) and  $R_{Q+1} = 1$  (one single antenna destination). The channel in the first MAC hop,  $\mathbf{H}_0$  is an  $R_2 \times 2$  matrix with the two columns  $\mathbf{H}_0^{[1]}, \mathbf{H}_0^{[2]}$  representing the channels from the two sources to the first hop relays. The channel in the last MAC hop  $\mathbf{H}_Q$  is a  $1 \times R_Q$  matrix representing the channel vector from the last hop relay transmitters to the single antenna destination. The total transmit power associated with the  $q^{th}$  hop is  $P_{q-1}^R$ . The two sources have transmit powers  $P_1$  and  $P_2$  with  $P_1 + P_2 = P = P_0^R$  as the first hop transmit power. For the  $q^{th}$  hop the receiving relay stage

has zero mean identity covariance AWGN denoted by  $\mathbf{N}_q$ .

For the dual BC the  $q^{\text{th}}$  hop experiences the channel  $\mathbf{H}_{Q+1-q}^\dagger$ , is associated with total transmit power  $P_{Q+1-q}^R$ , has the relay amplification matrix  $\mathbf{D}_{Q+1-q}^{b\dagger}$  and has zero mean unit covariance AWGN at the receivers as well.

1)  $MAC\left([\mathbf{H}_q]_0^Q, [\mathbf{D}_q^m]_1^Q, [P_q^R]_1^Q, P_1, P_2\right)$ : We refer to the MAC described above and shown in Fig. 5 as  $MAC\left([\mathbf{H}_q]_0^Q, [\mathbf{D}_q^m]_1^Q, [P_q^R]_1^Q, P_1, P_2\right)$ . We further characterize this MAC as follows.

The output signal for the MAC is:

$$y = \mathbf{H}_Q \mathbf{D}_Q^m \left( \cdots \left( \mathbf{H}_2 \mathbf{D}_2^m \left( \mathbf{H}_1 \mathbf{D}_1^m \left( \mathbf{H}_0^{[1]} x_1 + \mathbf{H}_0^{[2]} x_2 + \mathbf{N}_1 \right) + \mathbf{N}_2 \right) + \mathbf{N}_3 \right) \cdots + \mathbf{N}_Q \right) + n.$$

With noise power normalized to unity, the MAC output signal can be expressed as:

$$y = \frac{1}{\sqrt{\Delta^m}} \mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]} x_1 + \frac{1}{\sqrt{\Delta^m}} \mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]} x_2 + n \quad (56)$$

where

$$\Delta^m = 1 + \|\mathbf{H}_Q \mathbf{D}_Q^m\|^2 + \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m\|^2 + \cdots + \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \mathbf{H}_1 \mathbf{D}_1^m\|^2 \quad (57)$$

and the power constraints are:

$$\begin{aligned} P_0^R &= \mathbb{E}|x_1|^2 + \mathbb{E}|x_2|^2 = P_1 + P_2 = P \quad (58) \\ P_1^R &= \|\mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_1^m\|^2 \\ P_2^R &= \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m\|^2 + \|\mathbf{D}_2^m\|^2 \\ &\vdots \\ P_q^R &= \|\mathbf{D}_q^m \mathbf{H}_{q-1} \cdots \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_q^m \mathbf{H}_{q-1} \cdots \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 \\ &\quad + \sum_{j=1}^{q-1} \|\mathbf{D}_q^m \mathbf{H}_{q-1} \mathbf{D}_{q-1}^m \cdots \mathbf{H}_j \mathbf{D}_j^m\|^2 + \|\mathbf{D}_q^m\|^2, \quad q \in \{1, 2, \dots, Q\}. \quad (59) \end{aligned}$$

2)  $BC\left([\mathbf{H}_q^\dagger]_Q^0, [\mathbf{D}_q^{b\dagger}]_Q^1, [P_q^R]_Q^1, P_0^R = P\right)$ : The dual BC as shown in Fig. 6 is denoted as  $BC\left([\mathbf{H}_q^\dagger]_Q^0, [\mathbf{D}_q^{b\dagger}]_Q^1, [P_q^R]_Q^1\right)$ . The BC is further characterized as follows.

The output signals for the BC are:

$$\begin{aligned} y_1 &= \mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \left( \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \left( \cdots \left( \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \left( \mathbf{H}_Q^\dagger x + \mathbf{N}_Q \right) + \mathbf{N}_{Q-1} \right) \cdots + \mathbf{N}_2 \right) + \mathbf{N}_1 \right) + n_1, \\ y_2 &= \mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \left( \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \left( \cdots \left( \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \left( \mathbf{H}_Q^\dagger x + \mathbf{N}_Q \right) + \mathbf{N}_{Q-1} \right) \cdots + \mathbf{N}_2 \right) + \mathbf{N}_1 \right) + n_2. \end{aligned}$$

With noise power normalized to unity we have:

$$y_1 = \frac{1}{\sqrt{\Delta_1^b}} \mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger x + n_1, \quad (60)$$

$$y_2 = \frac{1}{\sqrt{\Delta_2^b}} \mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger x + n_2, \quad (61)$$

where

$$\Delta_1^b = 1 + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger}\|^2 + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger}\|^2 + \dots + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \dots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \quad (62)$$

$$\Delta_2^b = 1 + \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger}\|^2 + \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger}\|^2 + \dots + \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \dots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \quad (63)$$

and the relay power constraints for the dual BC are:

$$\begin{aligned} P_Q^R &= \mathbb{E}|x|^2 \\ P_{Q-1}^R &= \|\mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 P_Q^R + \|\mathbf{D}_Q^{b\dagger}\|^2 \\ P_{Q-2}^R &= \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 P_Q^R + \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 + \|\mathbf{D}_{Q-1}^{b\dagger}\|^2 \\ &\vdots \\ P_q^R &= \|\mathbf{D}_{q+1}^{b\dagger} \mathbf{H}_{q+1}^\dagger \dots \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 P_Q^R + \sum_{j=q}^{Q-2} \|\mathbf{D}_{q+1}^{b\dagger} \mathbf{H}_{q+1}^\dagger \mathbf{D}_{q+2}^{b\dagger} \dots \mathbf{H}_{j+1}^\dagger \mathbf{D}_{j+2}^{b\dagger}\|^2 + \|\mathbf{D}_{q+1}^{b\dagger}\|^2 \end{aligned}$$

for  $q \in \{0, 1, \dots, Q-1\}$ .

The following theorem states the duality result for the multiple hop AF relay MAC and BC with multiple antenna relays and single antenna source and destination nodes:

**Theorem 9:** For the  $(Q+1)$ -hop distributed multiple antenna AF relay MAC and BC described above and depicted in Fig. 5 and Fig. 6, given any set of BC relay amplification factors  $\mathbf{D}_q^b$  that satisfy the BC relay power constraints if we choose the MAC relay amplification factors as

$$\mathbf{D}_q^m = \kappa_q \mathbf{D}_q^{b\dagger}, \quad (64)$$

where  $\kappa_q$  are determined by the MAC relay power constraints, then

$$\mathcal{C}^{MAC} \left( [\mathbf{H}_q]_0^Q, [\mathbf{D}_q^m]_1^Q, [P_q^R]_1^Q, P_1, P_2 \right) \subset \mathcal{C}^{BC} \left( [\mathbf{H}_q]_Q^0, [\mathbf{D}_q^{b\dagger}]_Q^1, [P_q^R]_Q^1, P_0^R = P_1 + P_2 \right)$$

and

$$\begin{aligned} &\mathcal{C}^{BC} \left( [\mathbf{H}_q]_Q^0, [\mathbf{D}_q^{b\dagger}]_Q^1, [P_q^R]_Q^1, P_0^R = P \right) \\ &= \cup_{P_1, P_2 \geq 0, P_1 + P_2 = P} \mathcal{C}^{MAC} \left( [\mathbf{H}_q]_0^Q, [\mathbf{D}_q^m]_1^Q, [P_q^R]_1^Q, P_1, P_2 \right). \end{aligned} \quad (65)$$

The proof is presented in Appendix VI.

## VII. CONCLUSION

We explored the capacity and duality aspects of multihop point to multipoint (BC) and multipoint to point (MAC) AF relay networks. The MAC-BC duality known for conventional one hop Gaussian channel was found to be applicable to multiple hop communication over AF relay networks where some of the relays may have multiple antennas. An interesting aspect of the AF relay MAC-BC duality is that the powers of the transmitter and the relays are shifted in the dual network. With distributed single antenna relay nodes we determined the optimal relay scaling factors for the entire rate region of the two

hop relay multiple access channel. Closed form expressions were found for the sum rate and individual maximum rates while simultaneous equations were found that can be solved to determine any rate pair on the boundary of the relay MAC. The rate region of the AF relay BC was evaluated using duality as the union of the AF relay MAC rate regions over different power splits between user 1 and user 2 while keeping the total power constant and equal to the total relay transmit power on the BC.

We conclude with a word about the generality of the results. While the duality can be extended to multiple users as well, we have focused on the two user case to avoid the cumbersome notational aspects of considering multiple users in addition to multiple relays multiple hops and multiple antennas. The capacity characterization of the relay MAC for the complex case will also require phase optimizations of the relay scaling factors. Finally, we conjecture the duality results will hold for AF relay MIMO MAC-BC as well. However we do not expect the MIMO results to be trivial extensions of the results in this paper. The MIMO relay MAC-BC duality is especially challenging as the structure of the transmitted signal covariance matrix on the dual MAC and BC will be different.

#### APPENDIX I PROOF OF THEOREM 3

Consider the normalized MAC characterization

$$y' = \sqrt{\frac{\text{SNR}_1}{P_1}} x_1 + \sqrt{\frac{\text{SNR}_2}{P_2}} x_2 + n', \quad (66)$$

where  $\sqrt{\text{SNR}_1} = \sqrt{P_1 \frac{\sum_{k=1}^R g_k d_k f_k^{[1]}}{1 + \sum_{k=1}^R d_k^2 g_k^2}}$  and  $\sqrt{\text{SNR}_2} = \sqrt{P_2 \frac{\sum_{k=1}^R g_k d_k f_k^{[2]}}{1 + \sum_{k=1}^R d_k^2 g_k^2}}$  have been defined as such to allow compact notation.

Characterizing the rate region of a two user MAC is equivalent to an optimization problem where a weighted sum of rates is maximized, i.e.  $\max \mu_1 R_1 + \mu_2 R_2$ . Without loss of generality let us assume  $\mu_1 \geq \mu_2$ . For every choice of the relay amplification matrix  $\mathbf{D}$  we obtain a scalar Gaussian MAC for which the capacity region is a pentagon and this rate pair is maximized for the corner point:

$$R_1 = \log(1 + \text{SNR}_1), R_2 = \log\left(1 + \frac{\text{SNR}_2}{1 + \text{SNR}_1}\right). \quad (67)$$

Thus, we need to solve the following optimization problem:

$$\max_{d_1, \dots, d_R} \mu_1 R_1 + \mu_2 R_2 = \max_{\mathbf{D}} (\mu_1 - \mu_2) \log(1 + \text{SNR}_1) + \mu_2 \log(1 + \text{SNR}_1 + \text{SNR}_2) \quad (68)$$

$$= \max_{\mathbf{D}} \mu_1' \log(1 + \text{SNR}_1) + \mu_2 \log(1 + \text{SNR}_1 + \text{SNR}_2), \quad (69)$$

where  $\mu_2$  and  $\mu_1' = \mu_1 - \mu_2$  are both positive. Thus, we need to maximize

$$\max_{\mathbf{D}} (1 + \text{SNR}_1)^{\mu_1'} (1 + \text{SNR}_1 + \text{SNR}_2)^{\mu_2}, \quad (70)$$

subject to the power constraints (27)-(29).

We start with the Lagrangian formulation

$$L(\mathbf{D}, \lambda) = \left(1 + \frac{P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})^2}{1 + \sum_{k=1}^R d_k^2 g_k^2}\right)^{\mu_1} \left(1 + \frac{P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})^2 + P_2(\sum_{k=1}^R d_k g_k f_k^{[2]})^2}{1 + \sum_{k=1}^R d_k^2 g_k^2}\right)^{\mu_2} - \lambda \left[ \sum_{k=1}^R d_k^2 \left(1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2}\right) - P_R \right]. \quad (71)$$

Setting the derivative  $\frac{\partial L(\mathbf{D}, \lambda)}{\partial d_i}$  to zero yields the KKT characterization of optimal  $d_i$ :

$$\begin{aligned} & \mu_2(1 + \text{SNR}_1^*)^{\mu_1} (1 + \text{SNR}_1^* + \text{SNR}_2^*)^{\mu_2 - 1} \left[ \frac{2P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})g_i f_i^{[1]} + 2P_2(\sum_{k=1}^R d_k g_k f_k^{[2]})g_i f_i^{[2]}}{1 + \sum_{k=1}^R d_k^2 g_k^2} \right. \\ & \left. - \frac{P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})^2 + P_2(\sum_{k=1}^R d_k g_k f_k^{[2]})^2}{(1 + \sum_{k=1}^R d_k^2 g_k^2)^2} 2g_i^2 d_i \right] + \mu_1'(1 + \text{SNR}_1^*)^{\mu_1 - 1} (1 + \text{SNR}_1^* + \text{SNR}_2^*)^{\mu_2} \\ & \left[ \frac{2P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})g_i f_i^{[1]}}{1 + \sum_{k=1}^R d_k^2 g_k^2} - \frac{P_1(\sum_{k=1}^R d_k g_k f_k^{[1]})^2}{(1 + \sum_{k=1}^R d_k^2 g_k^2)^2} 2g_i^2 d_i \right] = \lambda 2d_i \left(1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2}\right). \quad (72) \end{aligned}$$

Absorbing the constants into  $\lambda$  we have

$$\begin{aligned} & \mu_2(1 + \text{SNR}_1^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) g_i f_i^{[1]} + P_2 \left( \sum_{k=1}^R d_k g_k f_k^{[2]} \right) g_i f_i^{[2]} - (\text{SNR}_1^* + \text{SNR}_2^*) g_i^2 d_i \right] \\ & + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) g_i f_i^{[1]} - \text{SNR}_1^* g_i^2 d_i \right] = \lambda d_i \left(1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2}\right). \end{aligned}$$

$$\begin{aligned} \Rightarrow & \mu_2(1 + \text{SNR}_1^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right)^2 + P_2 \left( \sum_{k=1}^R d_k g_k f_k^{[2]} \right)^2 - (\text{SNR}_1^* + \text{SNR}_2^*) \sum_{k=1}^R g_k^2 d_k^2 \right] \\ & + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right)^2 - \text{SNR}_1^* \sum_{k=1}^R g_k^2 d_k^2 \right] = \lambda P_R. \quad (73) \end{aligned}$$

$$\Rightarrow \mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)\text{SNR}_1^* = \lambda P_R. \quad (74)$$

$$\begin{aligned} \Rightarrow & d_i \left[ \frac{\mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)\text{SNR}_1^*}{P_R} \left(1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2}\right) \right. \\ & \left. + \mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*)g_i^2 + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)\text{SNR}_1^*g_i^2 \right] \\ & = \mu_2(1 + \text{SNR}_1^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) g_i f_i^{[1]} + P_2 \left( \sum_{k=1}^R d_k g_k f_k^{[2]} \right) g_i f_i^{[2]} \right] \\ & \quad + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) g_i f_i^{[1]}. \quad (75) \end{aligned}$$

$$\begin{aligned} \Rightarrow & d_i \left[ \frac{\mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)(\text{SNR}_1^*)}{P_R} \left(1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2\right) \right] \\ & = \mu_2(1 + \text{SNR}_1^*) \left[ P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) g_i f_i^{[1]} + P_2 \left( \sum_{k=1}^R d_k g_k f_k^{[2]} \right) g_i f_i^{[2]} \right] + \end{aligned}$$

$$\mu'_1(1 + \text{SNR}_1^* + \text{SNR}_2^*)P_1\left(\sum_{k=1}^R d_k g_k f_k^{[1]}\right)g_i f_i^{[1]}. \quad (76)$$

$$\begin{aligned} \Rightarrow d_i \frac{\mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu'_1(1 + \text{SNR}_1^* + \text{SNR}_2^*)(\text{SNR}_1^*)}{P_R} = \\ [\mu'_1(1 + \text{SNR}_1^* + \text{SNR}_2^*) + \mu_2(1 + \text{SNR}_1^*)]P_1\left(\sum_{k=1}^R d_k g_k f_k^{[1]}\right) \frac{g_i f_i^{[1]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \\ + \mu_2(1 + \text{SNR}_1^*)P_2\left(\sum_{k=1}^R d_k g_k f_k^{[2]}\right) \frac{g_i f_i^{[2]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2}. \end{aligned} \quad (77)$$

$$\Rightarrow \mathbf{D} = \mathbf{G} \left( c_1 P_1 \mathbf{F}^{[1]} + c_2 P_2 \mathbf{F}^{[2]} \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1} \quad \text{where } c_1, c_2 \geq 0, \quad (78)$$

$$\Rightarrow \mathbf{D} = \sqrt{c_1^2 + c_2^2} \mathbf{G} \left( \frac{c_1}{\sqrt{c_1^2 + c_2^2}} P_1 \mathbf{F}^{[1]} + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} P_2 \mathbf{F}^{[2]} \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1}. \quad (79)$$

Defining  $\gamma = \sqrt{c_1^2 + c_2^2}$  and  $\tan \theta = c_1/c_2$ , the optimal  $\mathbf{D}$  for maximizing  $\mu_1 R_1 + \mu_2 R_2$  can be expressed in the form:

$$\mathbf{D} = \gamma \mathbf{G} \left( P_1 \mathbf{F}^{[1]} \sin \theta + P_2 \mathbf{F}^{[2]} \cos \theta \right) \left( I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2 \right)^{-1}. \quad (80)$$

Note that the constant  $\gamma$  can be evaluated from the power constraint as in Theorem 3. ■

## APPENDIX II

### PROOF OF THEOREM 4

We are interested in the case  $\mu_1 = 1, \mu_2 = 0$ . Substituting these values into the optimal  $\mathbf{D}$  characterization of (77) we have

$$d_i \frac{\text{SNR}_1^*}{P_R} = P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) \frac{g_i f_i^{[1]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2}, \quad (81)$$

which corresponds to  $\theta = \pi/2$  and leads to

$$\text{SNR}_1^* = P_1 P_R \sum_{i=1}^R \frac{g_i^2 f_i^{[1]2}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2}, \quad (82)$$

which gives  $C_1^{10} = \log(1 + \text{SNR}_1^*)$ . In order to find  $C_2^{10}$ , we need  $\text{SNR}_2^*$ . From the definition,

$$\frac{\text{SNR}_2^*}{\text{SNR}_1^*} = \frac{P_2 (\sum_{k=1}^R d_k g_k f_k^{[2]})^2}{P_1 (\sum_{k=1}^R d_k g_k f_k^{[1]})^2}. \quad (83)$$

From (81) we have

$$\frac{(\sum_{k=1}^R d_k g_k f_k^{[2]})}{(\sum_{k=1}^R d_k g_k f_k^{[1]})} \text{SNR}_1^* = P_1 P_R \sum_{i=1}^R \frac{g_i^2 f_i^{[1]} f_i^{[2]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \quad (84)$$

$$\Rightarrow \text{SNR}_2^* = \frac{(\sum_{k=1}^R d_k g_k f_k^{[2]})^2}{(\sum_{k=1}^R d_k g_k f_k^{[1]})^2} \text{SNR}_1^* \frac{P_2}{P_1} \quad (85)$$

$$\Rightarrow \text{SNR}_2^* = P_2 P_R \frac{\left( \sum_{i=1}^R \frac{g_i^2 f_i^{[1]} f_i^{[2]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \right)^2}{\sum_{i=1}^R \frac{g_i^2 f_i^{[1]2}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2}}. \quad (86)$$

Substituting into (67) we have the expression for  $C_2^{10}$ . ■

### APPENDIX III

#### PROOF OF THEOREM 5

For sum rate we have  $\mu_1 = \mu_2 = 1$ , i.e.  $\mu_1' = 0, \mu_2 = 1$ . Substituting these values into (77) and defining  $\text{SNR}^* = \text{SNR}_1^* + \text{SNR}_2^*$ , we have

$$d_k \frac{\text{SNR}^*}{P_R} = \frac{P_1 \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right) g_k f_k^{[1]} + P_2 \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right) g_k f_k^{[2]}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2} \quad (87)$$

$$\Rightarrow \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right) \frac{\text{SNR}^*}{P_R} = P_1 A_{11} \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right) + P_2 A_{12} \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right) \quad (88)$$

$$\text{and} \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right) \frac{\text{SNR}^*}{P_R} = P_1 A_{12} \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right) + P_2 A_{22} \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right) \quad (89)$$

$$\Rightarrow \frac{\text{SNR}^*}{P_R} = P_1 A_{11} + P_2 A_{12} \frac{\left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)}{\left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)} \quad (90)$$

$$\text{and} \frac{\text{SNR}^*}{P_R} = P_1 A_{12} \frac{\left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)}{\left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)} + P_2 A_{22}. \quad (91)$$

Eliminating the factor  $\frac{(\sum_{j=1}^R g_j d_j f_j^{[1]})}{(\sum_{i=1}^R g_i d_i f_i^{[2]})}$  from the last two equations leads to the quadratic equation:

$$\left( \frac{\text{SNR}^*}{P_R} \right)^2 - (P_1 A_{11} + P_2 A_{22}) \frac{\text{SNR}^*}{P_R} + P_1 P_2 (A_{11} A_{22} - A_{12}^2) = 0 \quad (92)$$

$$\Rightarrow \frac{\text{SNR}^*}{P_R} = \frac{P_1 A_{11} + P_2 A_{22} \pm \sqrt{(P_1 A_{11} + P_2 A_{22})^2 - 4 P_1 P_2 (A_{11} A_{22} - A_{12}^2)}}{2}. \quad (93)$$

The smaller root of the quadratic equation represents the *minimum* value of the objective function, while the larger root is the desired *maximum* value, which gives us the sum rate expression of Theorem 5.

From definition,  $\tan \theta = \frac{(\sum_{i=1}^R g_i d_i f_i^{[2]})}{(\sum_{j=1}^R g_j d_j f_j^{[1]})}$  and its value is computed from (90).

Finally, we obtain the sum-rate optimal rate pairs by finding out  $\text{SNR}_1^*$  and  $\text{SNR}_2^*$  as follows.

$$\text{SNR}^* = \frac{P_1 \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2 + P_2 \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2}{1 + \sum_{l=1}^R d_l^2 g_l^2}, \quad (94)$$

$$\text{SNR}_1^* = \frac{P_1 \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2}{1 + \sum_{l=1}^R d_l^2 g_l^2}, \quad \text{SNR}_2^* = \frac{P_2 \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2}{1 + \sum_{l=1}^R d_l^2 g_l^2} \quad (95)$$

$$\text{Define } \beta = \frac{\text{SNR}_1^*}{\text{SNR}^*} \text{ so } \text{SNR}_1^* = \beta \text{SNR}^*, \text{SNR}_2^* = (1 - \beta) \text{SNR}^* \quad (96)$$

$$\Rightarrow \beta = \frac{P_1 \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2}{P_1 \left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2 + P_2 \left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2} = \frac{P_1 \frac{\left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2}{\left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2}}{P_1 \frac{\left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2}{\left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2} + P_2} \quad (97)$$

Substituting the value of  $\frac{\left( \sum_{j=1}^R g_j d_j f_j^{[1]} \right)^2}{\left( \sum_{i=1}^R g_i d_i f_i^{[2]} \right)^2}$  from (90) we have the definition of  $\beta$  as stated in Theorem 5. ■

#### APPENDIX IV

#### PROOF OF THEOREM 6

Continuing from (77),

$$\begin{aligned} d_i \frac{\mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)(\text{SNR}_1^*)}{P_R} = \\ [\mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) + \mu_2(1 + \text{SNR}_1^*)] P_1 \left( \sum_{k=1}^R d_k g_k f_k^{[1]} \right) \frac{g_i f_i^{[1]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \\ + \mu_2(1 + \text{SNR}_1^*) P_2 \left( \sum_{k=1}^R d_k g_k f_k^{[2]} \right) \frac{g_i f_i^{[2]}}{1 + P_1 f_i^{[1]2} + P_2 f_i^{[2]2} + P_R g_i^2} \end{aligned} \quad (98)$$

Multiplying both sides of the equation with  $g_i f_i^{[1]}$  and summing up all the equations for  $i = 1$  to  $i = R$ , we have:

$$\begin{aligned} \mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)(\text{SNR}_1^*) = \\ \mu_2(1 + \text{SNR}_1^*)(P_R P_1 A_{11} + P_R P_2 A_{12}/\alpha) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) P_R P_1 A_{11}, \end{aligned} \quad (99)$$

where  $\alpha = \frac{\left( \sum d g f^{[1]} \right)}{\left( \sum d g f^{[2]} \right)} = \sqrt{\frac{P_1 \text{SNR}_1^*}{P_2 \text{SNR}_2^*}}$ . Similarly,

$$\begin{aligned} \mu_2(1 + \text{SNR}_1^*)(\text{SNR}_1^* + \text{SNR}_2^*) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*)(\text{SNR}_1^*) = \\ \mu_2(1 + \text{SNR}_1^*)(P_R P_1 A_{12} \alpha + P_R P_2 A_{22}) + \mu_1'(1 + \text{SNR}_1^* + \text{SNR}_2^*) P_R P_1 A_{12} \alpha. \end{aligned} \quad (100)$$

Simple algebraic manipulation of these two simultaneous equations gives us the result of Theorem 6. ■

## PROOF OF THEOREMS 7 AND 8

Starting with the normalized BC as in (39), without loss of generality we assume

$$\frac{[\text{Tr}(\mathbf{F}^{[1]}\mathbf{D}\mathbf{G})]^2 P}{\text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[1]2} + P_R\mathbf{G}^2)]} \geq \frac{[\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G})]^2 P}{\text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)]}. \quad (101)$$

Thus, user 1 is the stronger user in this degraded AWGN broadcast channel.

We wish to show that any rate pair  $(R_1, R_2)$  achievable in  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$  is also achievable in  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$ . We establish this by showing that both sum rate optimal corner points (corresponding to different decoding orders) of the MAC rate region pentagon are contained in the broadcast rate region.

Consider first the sum rate optimal corner point  $(R_1^{MAC}(1 \rightarrow 2), R_2^{MAC}(1 \rightarrow 2))$  on the boundary of the rate region for AF relay  $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$  that corresponds to the successive decoding of user 1 followed by user 2.

$$\begin{aligned} R_1^{MAC}(1 \rightarrow 2) &= \log \left( 1 + \frac{P_1 P_R [\text{Tr}(\mathbf{G}\mathbf{D}\mathbf{F}^{[1]})]^2}{\text{Tr}[\mathbf{D}^2(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)] + P_2 P_R [\text{Tr}(\mathbf{G}\mathbf{D}\mathbf{F}^{[2]})]^2} \right) \\ R_2^{MAC}(1 \rightarrow 2) &= \log \left( 1 + \frac{P_2 P_R [\text{Tr}(\mathbf{G}\mathbf{D}\mathbf{F}^{[2]})]^2}{\text{Tr}[\mathbf{D}^2(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)]} \right). \end{aligned} \quad (103)$$

We show that this rate pair  $(R_1^{MAC}(1 \rightarrow 2), R_2^{MAC}(1 \rightarrow 2))$  is achievable in  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$ . From the capacity of the scalar AWGN BC, we know every rate pair on the boundary of the capacity region of  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$  can be written as:

$$R_1^{BC}(2 \rightarrow 1) = \log \left( 1 + \frac{\alpha P_R P [\text{Tr}(\mathbf{F}^{[1]}\mathbf{D}\mathbf{G})]^2}{\text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[1]2} + P_R\mathbf{G}^2)]} \right), \quad (104)$$

$$R_2^{BC}(2 \rightarrow 1) = \log \left( 1 + \frac{(1 - \alpha) P_R P [\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G})]^2}{\alpha P_R P [\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G})]^2 + \text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)]} \right). \quad (105)$$

The notation  $(2 \rightarrow 1)$  above is due to the interpretation that this rate pair can be achieved by dirty paper coding of user 2 followed by user 1. Now, setting  $R_1^{MAC}(1 \rightarrow 2) = R_1^{BC}(2 \rightarrow 1)$  we obtain the value of  $\alpha$

$$\alpha = \frac{P_1 \text{Tr}[\mathbf{D}^2(I + P\mathbf{F}^{[1]2} + P_R\mathbf{G}^2)]}{P \text{Tr}[\mathbf{D}^2(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2} + P_R\mathbf{G}^2)] + P P_2 P_R [\text{Tr}(\mathbf{F}^{[2]}\mathbf{D}\mathbf{G})]^2}. \quad (106)$$

It is easily verified that  $0 \leq \alpha \leq 1$  so it is a feasible power split for the broadcast channel.

Similarly, setting  $R_2^{MAC}(1 \rightarrow 2) = R_2^{BC}(2 \rightarrow 1)$  we again solve for  $\alpha$  to obtain:

$$(1 - \alpha) = \frac{\alpha P_R P P_2 [\text{Tr}(\mathbf{F}^{[2]} \mathbf{D} \mathbf{G})]^2 + P_2 \text{Tr}[\mathbf{D}^2 (I + P \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]}{P \text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)]} \quad (107)$$

$$\begin{aligned} \Rightarrow \alpha & \left( P P_R P_2 [\text{Tr}(\mathbf{F}^{[2]} \mathbf{D} \mathbf{G})]^2 + P \text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)] \right) \\ & = P \text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)] - P_2 \text{Tr}[\mathbf{D}^2 (I + P \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)] \end{aligned} \quad (108)$$

$$\alpha = \frac{P_1 \text{Tr}[\mathbf{D}^2 (I + P \mathbf{F}^{[1]2} + P_R \mathbf{G}^2)]}{P \text{Tr}[\mathbf{D}^2 (I + P_1 \mathbf{F}^{[1]2} + P_2 \mathbf{F}^{[2]2} + P_R \mathbf{G}^2)] + P P_2 P_R [\text{Tr}(\mathbf{F}^{[2]} \mathbf{D} \mathbf{G})]^2}. \quad (109)$$

Thus, the same value of  $\alpha$  satisfies both  $R_1^{MAC}(1 \rightarrow 2) = R_1^{BC}(2 \rightarrow 1)$  and  $R_2^{MAC}(1 \rightarrow 2) = R_2^{BC}(2 \rightarrow 1)$ . Therefore, with this value of  $\alpha$ ,  $\text{BC}(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa \mathbf{D}, P_1 + P_2)$  can achieve the rate pair  $(R_1^{MAC}(1 \rightarrow 2), R_2^{MAC}(1 \rightarrow 2))$ .

We have shown the sum rate optimal corner point of the AF relay MAC rate region corresponding to the successive decoding order  $1 \rightarrow 2$  lies on the boundary of the AF relay BC. In order to show that the entire AF relay MAC rate region lies within the dual AF relay BC rate region we need to show that the other sum rate optimal corner point of the AF relay MAC rate region pentagon (corresponding to successive decoding order  $2 \rightarrow 1$ ) is also contained within the rate region of the AF relay BC. The proof is obtained exactly as above by switching indices 1 and 2 in equations (102)-(109). For the MAC, switching indices 1 and 2 gives us the sum rate optimal corner point  $R_1^{MAC}(2 \rightarrow 1), R_2^{MAC}(2 \rightarrow 1)$  that corresponds to successive decoding of user 2 followed by user 1. For the BC, switching indices 1 and 2 gives us the rate pair  $R_1^{BC}(1 \rightarrow 2), R_2^{BC}(1 \rightarrow 2)$  that is achieved with dirty paper encoding of user 1 followed by user 2. The equality of  $R_1^{MAC}(2 \rightarrow 1), R_2^{MAC}(2 \rightarrow 1)$  and  $R_1^{BC}(1 \rightarrow 2), R_2^{BC}(1 \rightarrow 2)$  respectively is established by equations (102)-(109) with the indices 1 and 2 switched throughout. Note that with user 1 as the stronger user, only  $R_1^{BC}(2 \rightarrow 1), R_2^{BC}(2 \rightarrow 1)$  lies on the boundary of the BC capacity region while  $R_1^{BC}(1 \rightarrow 2), R_2^{BC}(1 \rightarrow 2)$  is contained inside the BC capacity region. Thus we have shown that the entire rate region of the AF relay MAC( $\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R$ ) is also achievable in AF relay BC( $\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa \mathbf{D}, P_1 + P_2$ ). This establishes Theorem 7.

Theorem 8 is also directly established from the above result. This is because we have shown not only that the dual BC achieves the rate pairs on the boundary of the MAC, but also that the rate pair  $R_1^{MAC}(1 \rightarrow 2), R_2^{MAC}(1 \rightarrow 2)$  is on the boundary of the rate region of the dual BC. In other words, we have shown above, that any choice of  $P_1, P_2$  on MAC( $\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R$ ) achieves a point *on the boundary* of the rate region of BC( $\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa \mathbf{D}, P_1 + P_2$ ). Moreover  $P_1, P_2$  are related to  $\alpha$  by a continuous function. Now, as we take  $P_1$  from 0 to  $P$  while keeping  $P_1 + P_2 = P$ , we trace the entire boundary of the capacity region of BC( $\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa \mathbf{D}, P_1 + P_2$ ). This implies the result of Theorem 8. ■

## PROOF OF THEOREM 9

We begin with the following useful observation.

**Property 1:** There is no loss of generality in the statement of Theorem 9 if all relay transmit powers are assumed equal to 1, i.e.  $P_1^R = P_2^R = \dots = P_Q^R = 1$  and  $P_0^R = P_1 + P_2 = 1$ .

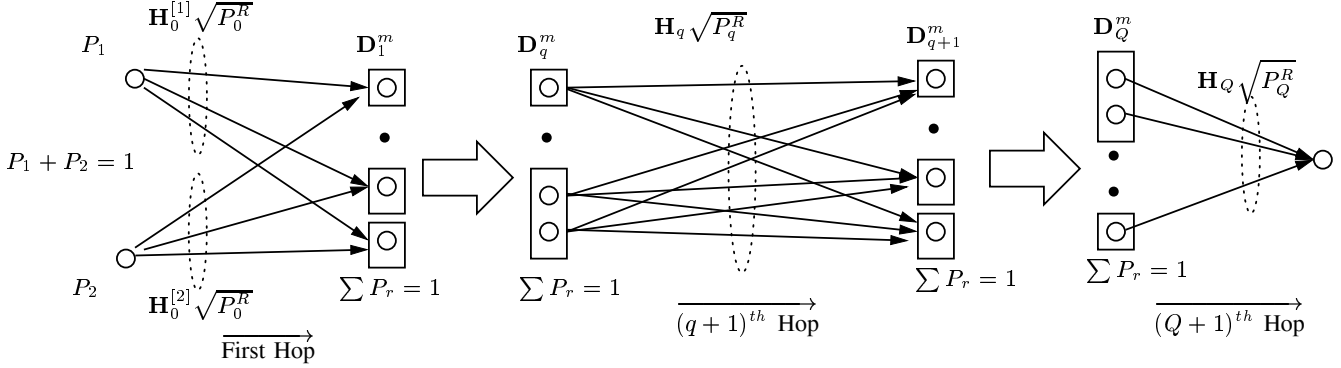


Fig. 7.  $(Q + 1)$ -hop MAC with normalized source and relay transmit powers

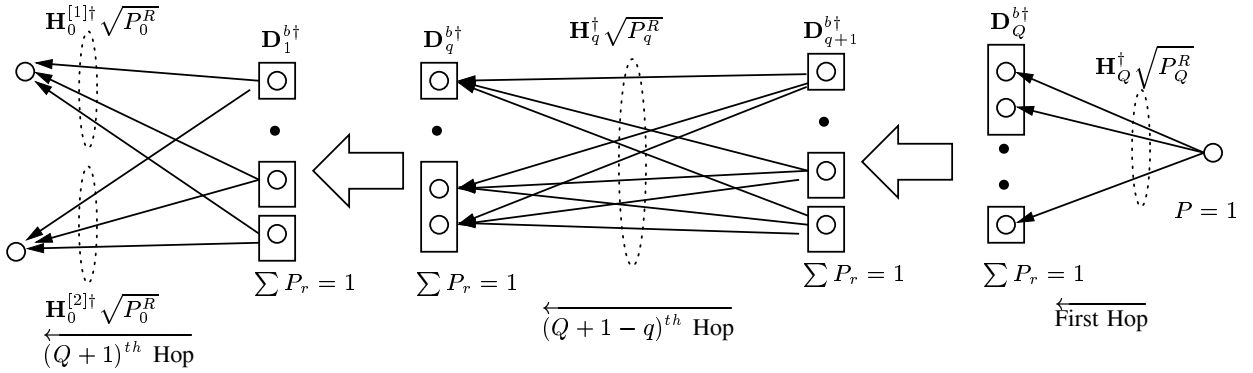


Fig. 8. Dual  $(Q + 1)$ -hop BC with normalized source and relay transmit powers

Property 1 is evident from Fig. 7 and Fig. 8. The power constraints can be absorbed into the channels by scaling the channel coefficients. The key to this property is that the scaled channels are still identical in the dual AF relay MAC and BC, i.e. the scaling factors are the same in the two directions. This is because the relay power constraints in the BC are shifted so that each channel experiences the same transmit power as in the dual MAC.

In light of this property, we will henceforth assume, without loss of generality that  $P_1^R = P_2^R = \dots = P_Q^R = 1$  and  $P_0^R = P_1 + P_2 = 1$ . Consider arbitrary but given feasible scaling matrices on the broadcast channel indicated as  $\mathbf{D}_1^b, \dots, \mathbf{D}_Q^b$ . Feasibility means that the BC relay power constraints are satisfied. Scaling matrices on the MAC are defined according to the statement of Theorem 9 as  $\mathbf{D}_q^m = \kappa_q \mathbf{D}_q^{b\dagger}$ . The constants  $\kappa_q$  are real, non-negative and uniquely determined by the MAC relay power constraints.

Proceeding as in the proof of Theorem 7, let us assume, again without loss of generality that user 1 is the stronger user on the resulting BC., i.e.,

$$\frac{\|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2}{\Delta_1^b} \geq \frac{\|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2}{\Delta_2^b}. \quad (110)$$

Following the proof of Theorem 7 we need to show that the following rate pair

$$R_1^{MAC} = \log \left( 1 + \frac{P_1 \|\mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2}{\Delta^m + P_2 \|\mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2} \right), \quad (111)$$

$$R_2^{MAC} = \log \left( 1 + \frac{P_2 \|\mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2}{\Delta^m} \right), \quad (112)$$

is achievable in the dual BC and that it lies *on the boundary* of the dual BC capacity region for the given relay amplification matrices.

Rate pairs on the boundary of the degraded dual BC with user 1 stronger than user 2 are given by:

$$R_1^{BC} = \log \left( 1 + \frac{\alpha}{\Delta_1^b} \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 \right), \quad (113)$$

$$R_2^{BC} = \log \left( 1 + \frac{(1-\alpha) \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2}{\Delta_2^b + \alpha \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2} \right). \quad (114)$$

Setting  $R_1^{MAC} = R_1^{BC}$  and solving for  $\alpha$  we obtain the value,

$$\begin{aligned} \alpha_1 &= \frac{\Delta_1^b P_1 \left( \prod_{q=1}^Q \kappa_q^2 \right)}{\Delta^m + P_2 \|\mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2} \\ &= \frac{\Delta_1^b P_1}{\Delta^m / \left( \prod_{q=1}^Q \kappa_q^2 \right) + P_2 \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2}. \end{aligned} \quad (115)$$

Setting  $R_2^{MAC} = R_2^{BC}$  and solving for  $\alpha$  we obtain the value,

$$\begin{aligned} \alpha_2 &= \frac{\Delta^m - \Delta_2^b P_2 \left( \prod_{q=1}^Q \kappa_q^2 \right)}{\Delta^m + P_2 \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2} \\ &= \frac{\Delta^m / \left( \prod_{q=1}^Q \kappa_q^2 \right) - \Delta_2^b P_2}{\Delta^m / \left( \prod_{q=1}^Q \kappa_q^2 \right) + P_2 \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2}. \end{aligned} \quad (116)$$

To complete the proof we need to show that the same value of  $\alpha$  satisfies both  $R_1^{MAC} = R_1^{BC}$  and  $R_2^{MAC} = R_2^{BC}$ . This would imply both that the MAC rate pair is achievable in the dual BC and also imply that it is on the boundary of the dual BC capacity region. In other words it remains to be shown that:

$$\Delta^m / \left( \prod_{q=1}^Q \kappa_q^2 \right) = P_1 \Delta_1^b + P_2 \Delta_2^b. \quad (117)$$

This is the most challenging part of the proof of the duality relationship. The remainder of this section is devoted to establishing (117). Part of the challenge is the unwieldy nature of the expressions involved. To allow compact expressions, we introduce the following shorthand notation:

$$\|\mathbf{A}\overline{\mathbf{B}}\mathbf{C}\|^2 \triangleq \|\mathbf{ABC}\|^2 + \|\mathbf{A}\|^2\|\mathbf{C}\|^2. \quad (118)$$

Thus, the  $\overline{\mathbf{B}}$  notation gives rise to two terms, one with  $\mathbf{B}$  present and one with  $\mathbf{B}$  absent so that the chain is broken into product of the norms of the LHS and the RHS of  $\mathbf{B}$ .

Note that the notation follows an associative property, i.e.,

$$\|\mathbf{A}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}\mathbf{E}\|^2 = \|\mathbf{A}\overline{\mathbf{B}}(\mathbf{C}\overline{\mathbf{D}}\mathbf{E})\|^2 \quad (119)$$

$$= \|\mathbf{ABC}\overline{\mathbf{D}}\mathbf{E}\|^2 + \|\mathbf{A}\|^2\|\mathbf{C}\overline{\mathbf{D}}\mathbf{E}\|^2 \quad (120)$$

$$= \|\mathbf{ABCDE}\|^2 + \|\mathbf{ABC}\|^2\|\mathbf{E}\|^2 + \|\mathbf{A}\|^2\|\mathbf{CDE}\|^2 + \|\mathbf{A}\|^2\|\mathbf{C}\|^2\|\mathbf{E}\|^2 \quad (121)$$

$$= \|\mathbf{A}\overline{\mathbf{B}}\mathbf{CDE}\|^2 + \|\mathbf{A}\overline{\mathbf{B}}\mathbf{C}\|^2\|\mathbf{E}\|^2 \quad (122)$$

$$= \|(\mathbf{A}\overline{\mathbf{B}}\mathbf{C})\overline{\mathbf{D}}\mathbf{E}\|^2. \quad (123)$$

Thus, the two  $\overline{\mathbf{B}}$  and  $\overline{\mathbf{D}}$  terms give rise to four terms overall, as each of  $\mathbf{B}$  and  $\mathbf{D}$  can be present or absent.

Also note that  $\|\mathbf{A}\overline{\mathbf{B}}\|^2 = \|\mathbf{A}\overline{\mathbf{B}}\mathbf{I}\|^2 = \|\mathbf{AB}\|^2 + \|\mathbf{A}\|^2\|\mathbf{I}\|^2$ . If  $\mathbf{B}$  is a column vector then  $\mathbf{I}$  is just a  $1 \times 1$  identity matrix with  $\|\mathbf{I}\|^2 = 1$ . Using this notation we recursively manipulate the relay power constraints on the MAC into a suitable form.

For the sources,

$$1 = \underbrace{P_1 + P_2}_{E_0^m} \quad (124)$$

For the first set of relays,

$$\begin{aligned} 1 &= \|\mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_1^m\|^2 \\ &= \|\mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_1^m\|^2 E_0^m \\ &= \underbrace{\|\mathbf{D}_1^m \overline{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_1^m \overline{\mathbf{H}}_0^{[2]}\|^2 P_2}_{E_1^m}. \end{aligned} \quad (125)$$

For the second set of relays,

$$\begin{aligned} 1 &= \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m\|^2 + \|\mathbf{D}_2^m\|^2 \\ &= \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 + \|\mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m\|^2 E_0^m + \|\mathbf{D}_2^m\|^2 E_1^m \\ &= \underbrace{\|\mathbf{D}_2^m \overline{\mathbf{H}}_1 \mathbf{D}_1^m \overline{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_2^m \overline{\mathbf{H}}_1 \mathbf{D}_1^m \overline{\mathbf{H}}_0^{[2]}\|^2 P_2}_{E_2^m}. \end{aligned} \quad (126)$$

For the third set of relays,

$$\begin{aligned} 1 &= \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 \\ &\quad + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m\|^2 + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m\|^2 + \|\mathbf{D}_3^m\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2 \\
&\quad + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m \mathbf{H}_1 \mathbf{D}_1^m\|^2 E_0^m + \|\mathbf{D}_3^m \mathbf{H}_2 \mathbf{D}_2^m\|^2 E_1^m + \|\mathbf{D}_3^m\|^2 E_2^m \\
&= \underbrace{\|\mathbf{D}_3^m \bar{\mathbf{H}}_2 \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_3^m \bar{\mathbf{H}}_2 \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[2]}\|^2 P_2}_{E_3^m}. \tag{127}
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
1 &= \underbrace{\|\mathbf{D}_q^m \bar{\mathbf{H}}_{q-1} \cdots \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_q^m \bar{\mathbf{H}}_{q-1} \cdots \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[2]}\|^2 P_2}_{E_q^m}. \tag{128}
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
1 &= \underbrace{\|\mathbf{D}_Q^m \bar{\mathbf{H}}_{Q-1} \cdots \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\mathbf{D}_Q^m \bar{\mathbf{H}}_{Q-1} \cdots \mathbf{D}_2^m \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[2]}\|^2 P_2}_{E_Q^m}. \tag{129}
\end{aligned}$$

Notice that at the  $q^{\text{th}}$  relay stage, by appropriately substituting the power constraint expressions from all preceding relay stages, a homogeneous expression is obtained where every term contains  $\mathbf{D}_1^m, \dots, \mathbf{D}_q^m$ . For the  $q^{\text{th}}$  relay stage, the expression  $E_q^m$  contains  $2^q + 2^q$  terms as every  $\bar{\mathbf{H}}_i$  for  $i \in \{0, 1, \dots, q-1\}$  gives rise to two terms, one with  $\mathbf{H}_i$  present and one with  $\mathbf{H}_i$  absent.

Next we substitute from the power constraints into the expression for  $\Delta^m$  with the goal of obtaining a homogeneous expression where every term contains all the relay amplification matrices  $\mathbf{D}_1^m, \dots, \mathbf{D}_Q^m$ .

$$\begin{aligned}
\Delta^m &= 1 + \|\mathbf{H}_Q \mathbf{D}_Q^m\|^2 + \cdots + \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \mathbf{H}_2 \mathbf{D}_2^m\|^2 + \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \mathbf{H}_1 \mathbf{D}_1^m\|^2 \\
&= E_Q^m + \sum_{j=1}^Q \|\mathbf{H}_Q \mathbf{D}_Q^m \cdots \mathbf{H}_j \mathbf{D}_j^m\|^2 E_{j-1}^m \\
&= \|\bar{\mathbf{H}}_Q \mathbf{D}_Q^m \bar{\mathbf{H}}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[1]}\|^2 P_1 + \|\bar{\mathbf{H}}_Q \mathbf{D}_Q^m \bar{\mathbf{H}}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \bar{\mathbf{H}}_1 \mathbf{D}_1^m \bar{\mathbf{H}}_0^{[2]}\|^2 P_2 \\
&\quad - \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[1]}\|^2 P_1 - \|\mathbf{H}_Q \mathbf{D}_Q^m \mathbf{H}_{Q-1} \mathbf{D}_{Q-1}^m \cdots \mathbf{H}_1 \mathbf{D}_1^m \mathbf{H}_0^{[2]}\|^2 P_2. \tag{130}
\end{aligned}$$

Next, we manipulate the BC power constraints into the desired form. For the first hop relays on the dual BC,

$$\begin{aligned}
1 &= \|\mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 + \|\mathbf{D}_Q^{b\dagger}\|^2 \\
&= \underbrace{\|\mathbf{D}_Q^{b\dagger} \bar{\mathbf{H}}_Q^\dagger\|^2}_{E_Q^b}. \tag{131}
\end{aligned}$$

For the second hop relays,

$$\begin{aligned}
1 &= \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 + \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 + \|\mathbf{D}_{Q-1}^{b\dagger}\|^2 \\
&= \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 + \|\mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 + \|\mathbf{D}_{Q-1}^{b\dagger}\|^2 E_Q^b \\
&= \underbrace{\|\mathbf{D}_{Q-1}^{b\dagger} \bar{\mathbf{H}}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \bar{\mathbf{H}}_Q^\dagger\|^2}_{E_{Q-1}^b}. \tag{132}
\end{aligned}$$

For the third hop relays,

$$\begin{aligned}
1 &= \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 + \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \\
&\quad + \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger}\|^2 + \|\mathbf{D}_{Q-2}^{b\dagger}\|^2 \\
&= \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q^\dagger\|^2 + \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger} \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \\
&\quad + \|\mathbf{D}_{Q-2}^{b\dagger} \mathbf{H}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger}\|^2 E_Q^b + \|\mathbf{D}_{Q-2}^{b\dagger}\|^2 E_{Q-1}^b \\
&= \underbrace{\|\mathbf{D}_{Q-2}^{b\dagger} \overline{\mathbf{H}}_{Q-2}^\dagger \mathbf{D}_{Q-1}^{b\dagger} \overline{\mathbf{H}}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \overline{\mathbf{H}}_Q^\dagger\|^2}_{E_{Q-2}^b}. \tag{133}
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
1 &= \underbrace{\|\mathbf{D}_q^{b\dagger} \overline{\mathbf{H}}_q^\dagger \cdots \mathbf{D}_Q^{b\dagger} \overline{\mathbf{H}}_Q^\dagger\|^2}_{E_q^b} \tag{134}
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
1 &= \underbrace{\|\mathbf{D}_1^{b\dagger} \overline{\mathbf{H}}_1^\dagger \cdots \mathbf{D}_Q^{b\dagger} \overline{\mathbf{H}}_Q^\dagger\|^2}_{E_1^b}. \tag{135}
\end{aligned}$$

The homogeneous form of the power constraints is now used to convert  $\Delta_1^b, \Delta_2^b$  into homogeneous forms as well.

$$\begin{aligned}
\Delta_1^b &= 1 + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger}\|^2 + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger}\|^2 + \cdots + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \tag{136} \\
&= E_1^b + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger}\|^2 E_2^b + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger}\|^2 E_3^b + \cdots + \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger}\|^2 \\
&= \|\overline{\mathbf{H}}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \overline{\mathbf{H}}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \overline{\mathbf{H}}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \overline{\mathbf{H}}_Q\|^2 - \|\mathbf{H}_0^{[1]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q\|^2. \tag{137}
\end{aligned}$$

Similarly,

$$\Delta_2^b = \|\overline{\mathbf{H}}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \overline{\mathbf{H}}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \overline{\mathbf{H}}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \overline{\mathbf{H}}_Q\|^2 - \|\mathbf{H}_0^{[2]\dagger} \mathbf{D}_1^{b\dagger} \mathbf{H}_1^\dagger \mathbf{D}_2^{b\dagger} \cdots \mathbf{H}_{Q-1}^\dagger \mathbf{D}_Q^{b\dagger} \mathbf{H}_Q\|^2. \tag{138}$$

It is now easily verified that  $\Delta^m / \left(\prod_{q=1}^Q \kappa_q^2\right) = P_1 \Delta_1^b + P_2 \Delta_2^b$ . Thus, the same value of  $\alpha$  satisfies both  $R_1^{MAC} = R_1^{BC}$  and  $R_2^{MAC} = R_2^{BC}$  and the chosen boundary point of the MAC is also shown to be a boundary point of the dual BC. By the same arguments as the proof of Theorem 7, the duality relationship is established. ■

## REFERENCES

- [1] H. Viswanathan and S. Mukherjee, "Throughput-range tradeoff of wireless mesh backhaul networks," *IEEE Journal on Selected Areas in Communications*, vol. 24, pp. 593–602, March 2006.
- [2] H. Viswanathan and S. Mukherjee, "Performance of cellular networks with relays and centralized scheduling," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 2318–2328, Sept. 2005.
- [3] T. Tang, C. Chae, R. Heath Jr., and S. Cho, "On achievable sum rates of a multiuser MIMO relay channel," in *Proceedings of IEEE Int. Symp. Inform. Theory*, 2006.
- [4] B. Schein, "Distributed coordination in network information theory," *Ph.D. Thesis at MIT Cambridge*, pp. 64–68, August 2001.

- [5] J. N. Laneman and G. W. Wornell, "Exploiting Distributed Spatial Diversity in Wireless Networks," in *Proc. Allerton Conf. Commun., Contr., Computing, Illinois*, Oct 2000.
- [6] D. Chen and J. N. Laneman, "Modulation and demodulation for cooperative diversity in wireless systems," *To appear in IEEE Trans. Wireless Commun.*, 2005.
- [7] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inform. Theory*, vol. 50, no. 12, pp. 3062–3080, 2004.
- [8] T. M. Cover and A. El Gamaal, "Capacity Theorems for the relay channel," *IEEE Transactions on Information Theory*, vol. 25, no. 5, pp. 572–584, Sep 1979.
- [9] G. Kramer, M. Gastpar and P. Gupta, "Cooperative Strategies and Capacity Theorems for Relay Channels," *IEEE Transactions on Information Theory*, vol. 51, pp. 3037–3063, Sep. 2005.
- [10] K. Gomadam and S. Jafar, "Optimizing soft information in relay networks," in *Proceedings of Asilomar Conf. on Signals, Systems, & Comp.*, 2006.
- [11] M. Gastpar and M. Vetterli, "On the Capacity of Large Gaussian Relay Networks," *IEEE Transactions on Information Theory*, vol. 51, no.3, pp. 765–779, March 2005.
- [12] M. Gastpar and M. Vetterli, "On the capacity of wireless networks: the relay case," in *Proc. 2002 INFOCOM*.
- [13] S. Zahedi, M. Mohseni, A. El Gamal, "On the capacity of AWGN relay channels with linear relaying functions," in *Proc. 2004 IEEE Int. Symp. Info. Theory*.
- [14] A.F. Dana, R. Gowaikar, B. Hassibi, M. Effros and M. Medard, "Should we break a wireless network into subnetworks?," in *Allerton Conference on Communication, Control and Computing.*, 2003.
- [15] A. E. Gamal and N. Hassanpour, "Relay-without-delay," in *Proceedings of ISIT 2005*, 2005.
- [16] S. Serbetli and A. Yener, "Optimal power allocation for relay assisted F/TDMA ad hoc networks," in *Proceedings of International Conference on Wireless Networks, Communications and Mobile Computing, 2005*, vol. 2, 2005.
- [17] M. Chen, S. Serbetli, and A. Yener, "Distributed power allocation for parallel relay networks," in *Proceedings of Globecom 2005*, 2005.
- [18] K. Lee and A. Yener, "Iterative power allocation algorithms for amplify/estimate/compress-and-forward multi-band relay," in *Proceedings of Conference on Information Sciences and Systems, 2006*, 2006.
- [19] I. Maric and R. D. Yates, "Bandwidth and power allocation for cooperative strategies in Gaussian relay networks," in *Conference Record of the Thirty-Eighth Asilomar Conference on Signals, Systems and Computers, 2004*, vol. 2, 2004.
- [20] R. Nabar, H. Boelcskei, and F. Kneubuhler, "Fading relay channels: Performance limits and space-time signal design," *IEEE Journal on Selected Areas in Communications*, vol. 22, pp. 1099–1109, Aug 2004.
- [21] P. Larsson and H. Rong, "Large scale cooperative relaying network with optimal coherent combining under aggregate relay power constraints," *Future Telecommunications Conference, Beijing, China*, December 2003.
- [22] M. O. Hasna and M.-S. Alouini, "Optimal power allocation for relayed transmissions over rayleigh-fading channels," *IEEE Trans. Inform. Theory*, vol. 3, pp. 1999–2004, Nov. 2004.
- [23] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, pp. 388–404, August 2000.
- [24] N. Khajehnouri and A. H. Sayed, "A distributed MMSE relay strategy for wireless sensor networks," in *Proc. IEEE Workshop on Signal Processing Advances in Wireless Communications, NY*, Jun 2005.
- [25] Y. Liang and V. Veeravalli, "Cooperative relay broadcast channels," in *To appear*, 2005. Preprint available at <http://www.princeton.edu/~yingbinl/publication.html>.
- [26] S. Berger and A. Wittneben, "Cooperative Distributed Multiuser MMSE Relaying in Wireless Ad Hoc Networks," in *Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA*, Oct 2005.
- [27] I. Maric and R. Yates, "Cooperative multicast for maximum network lifetime," *IEEE J. Select. Areas Commun.*, vol. 23, Jan 2005.
- [28] I. Maric and R. Yates, "Efficient multihop broadcast for wideband systems," *IEEE J. Select. Areas Commun.*, vol. 22, Aug 2004.

- [29] A. Host-Madsen, "Capacity bounds for cooperative diversity," *IEEE Trans. Inform. Theory*, vol. 52, pp. 1522–1544, April 2006.
- [30] K. Azarian, H. El Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Transactions on Information Theory*, vol. 51, pp. 4152–4172, Dec. 2005.
- [31] A. Dana and B. Hassibi, "On the power efficiency of sensory and ad hoc wireless networks," *IEEE Trans. Inform. Theory*, vol. 52, pp. 2890–2914, July 2006.

**Syed Ali Jafar** (S' 99- M'04) received the B. Tech. degree in Electrical Engineering from the Indian Institute of Technology (IIT), Delhi, India in 1997, the M.S. degree in Electrical Engineering from California Institute of Technology (Caltech), Pasadena USA in 1999, and the Ph.D. degree in Electrical Engineering from Stanford University, Stanford, CA USA in 2003. He was a summer intern in the Wireless Communications Group of Lucent Bell Laboratories, Holmdel, NJ, in 2001. He was an engineer in the Satellite Networks Division of Hughes Software Systems from 1997-1998 and a senior engineer at Qualcomm Inc., San Diego, CA in 2003. He is currently an Assistant Professor in the Department of Electrical Engineering and Computer Science at the University of California Irvine, Irvine, CA USA. His research interests include multiuser information theory and wireless communications.

Dr. Jafar received the NSF CAREER award in 2006. He is the recipient of the 2006 UC Irvine Engineering Faculty of the Year award for excellence in teaching. Dr. Jafar serves as the Editor for Wireless Communication Theory and CDMA for the *IEEE Transactions on Communications*.

**Krishna Srikanth Gomadam** (S' 05) received the B. Tech. degree in Electronics Engineering from Madras Institute of Technology, Anna University, in 2004, and the M.S. degree in Electrical Engineering from the University of California Irvine in 2007. He is currently working toward the Ph.D. degree at the same university. He was with the AU-KBC Research Center, Chennai, India, during the summers of 2003 and 2004.

Mr. Gomadam is a recipient of the UC Irvine CPCC graduate fellowship for the years 2005 and 2006. He received the UC Irvine EECS department's best student paper award for 2007. His research interests include cooperative communications and multi-user relay networks.

**Chiachi Huang** received the B.S. degree in Electrical Engineering from Yuan Ze University, Chungli, Taiwan in 1999 and the M.S. degree in communications engineering from National Tsing Hua University, Hsingchu, Taiwan in 2001. He is currently working toward the Ph.D. degree in Electrical Engineering at the University of California Irvine. His research interests include wireless communications and mesh networks.